APPLICATION OF THE TODD-COXETER COSET ENUMERATION ALGORITHM

Colin M. Campbell

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Abstract. This thesis is concerned with a topic in combinatorial group theory and, in particular, with a study of some groups with finite presentations. After preliminary definitions and theorems we describe the Todd-Coxeter coset enumeration algorithm and the modified Todd-Coxeter algorithm which shows that, given a finitely generated subgroup H of finite index in a finitely presented group G, we can find a presentation for H. We then give elementary examples illustrating the algorithms and include a discussion on the computer programmes that are to be used.

In the main part of the thesis we investigate two classes of cyclically presented groups. Suppose

$$G_n(w) \cong \langle a_1, a_2, \ldots, a_n \mid w_1 = 1, w_2 = 1, \ldots, w_n = 1 \rangle$$

where $w_1 = w$ is a word in $a_1, a_2, \ldots, a_n$ and w_{i+1} is obtained from w_i by applying the permutation $(1\ 2\ \ldots\ n)$ to the suffices of the a's. The first class we investigate are the groups

$$\underset{\sim}{G}(\ell,m,n) \cong \langle a,b \mid ab^m a^{-1} b^{-1} a^{-n} b^{1-\ell} = 1,\ ba^m b^{-1} a^{-1} b^{-n} a^{1-\ell} = 1 \rangle,$$

that is the groups $\underset{\sim}{G}(\ell,m,n)$ are groups of type $G_2(w)$. Secondly we investigate the Fibonacci-type groups $\underset{\sim}{H}(r,n,k,s,h)$ obtained when, for some integers $r,s,h \geq 1$, $k \geq 0$, the word w is given by

$$w = a_h a_{2h} \ldots a_{rh} (a_{rh+k}\, a_{(r+1)h+k} \ldots a_{(r+s-1)h+k})^{-1},$$

Fibonacci groups being the special case given by $k = s = h = 1$. For both classes we begin by giving some homomorphisms and isomorphisms that may be obtained. We show, using the Todd-Coxeter algorithm when appropriate, that the six groups $\underset{\sim}{G}(2,2,3)$, $\underset{\sim}{G}(2,2,-3)$, $\underset{\sim}{G}(-1,-1,4)$, $\underset{\sim}{G}(2,3,-2)$, $\underset{\sim}{G}(-2,2,-1)$ and $\underset{\sim}{G}(-2,3,1)$ are finite non-metacyclic groups of deficiency zero, having orders $2^{15}.3^3$, $2^8.3^3$, $2^9.3.5$, $2^3.3^3.7$, $2^3.3.5.11$ amd $2^3.3^6$ respectively. We also show that the groups $\underset{\sim}{G}(1-n, 6, n)$ where $n \equiv 1 \bmod 5$ give an infinite series of non-metacyclic groups.

We consider the structure of the non-metacyclic groups $\mathbf{H}(3,6,1,1,1)$ and $\mathbf{H}(3,6,5,1,2)$ both of order 1512, showing that neither is isomorphic to $\mathbf{G}(2, 3, -2)$ another non-metacyclic group of order 1512. In a paper on 'the Fibonacci groups' (Proc. London Math. Soc. 29 (1974), 577-592) D.L. Johnson, J.W. Wamsley and D. Wright pose two questions relating to the Fibonacci groups for the case $r \equiv 1 \bmod n$, namely to find 2-generator 2-relation presentations for them and also their orders. We answer these questions and generalise the results to the class $\mathbf{H}(r,n,k,s,1)$ where it is shown that $\mathbf{H}(r,n,k,s,1)$ is metacyclic if (i) $r \equiv s \bmod n$, (ii) $(r,n) = 1$, (iii) $(r + k - 1, n) = 1$, and a 2-generator 2-relation presentation is found for these groups. Further if (iv) $(r,s) = 1$, then we show that $\mathbf{H}(r,n,k,s,1)$ is a finite metacyclic group of order $r^n - s^n$. A possible generalisation to the groups $\mathbf{H}(r,n,k,s,h)$ is considered. Finally the metacyclic groups $\mathbf{H}(r,4,1,2,1)$, r odd are discussed.

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by

Colin M. Campbell

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Department of Pure Mathematics,
University of St. Andrews.

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Declaration

I declare that the following thesis is a record of research work carried out by me, that the thesis is my own composition, and that it has not been presented previously in application for a higher degree.

Declaration

I declare that, being a member of the teaching staff of the University, I was admitted in October 1965 under University Court Ordinance LXXIX (St. Andrews No.16) as a part-time Research Student in the Department of Mathematics.

I certify that Colin M. Campbell has satisfied the conditions of the Resolution and Regulations and is thus qualified to submit the accompanying thesis in application for the degree of doctor of philosophy.

Preface

I should like to express my thanks to my supervisor, Dr. Robertson, whose help and encouragement I have greatly appreciated throughout the course of this work. Further I should like to thank Dr. Beetham for many helpful conversations and also Professor Moser of McGill University who first introduced me to the ideas of coset enumeration. I should also like to acknowledge a grant from the Research Fund of the University of St. Andrews towards the cost of the preparation and production of this thesis. Finally I should like to thank the secretaries of the Mathematical Institute for the care with which they have typed this thesis.

CONTENTS

CHAPTER 1

AIM OF THESIS AND PRELIMINARY DEFINITIONS

1.1 Introduction.

In 1936 J. A. Todd and H. S. M. Coxeter (42) described an algorithm for enumerating the cosets of a finitely generated subgroup of finite index in a finitely generated finitely presented group. Their paper was based on earlier work by, among others, E. H. Moore (40). The algorithm is suitable for computer implementation and an early programme was written by Haselgrove at Cambridge in 1953. Many other mathematicians have used the algorithm or have written programmes for the algorithm. We mention here work by Beetham (1), Moser (41) and Trotter (43). Several authors (Beetham and Campbell (2), Benson and Mendelsohn (3), Campbell (7), Cannon (18), (19), Leech (32), (33) and Mendelsohn (36), (37))have discussed modifications to the algorithm and it is shown in (3), (7), (19), (33), (38) that this algorithm may be used to give a presentation of the subgroup in terms of the given subgroup generators.

1.2 Aim of thesis.

The aim of this thesis is to show how the Todd-Coxeter coset enumeration algorithm and the modified Todd-Coxeter algorithm may be used in the study of two classes of cyclically presented groups, the class $\underset{\sim}{G}(l,m,n)$ discussed in chapter 3 and a class of generalised Fibonacci groups $\underset{\sim}{H}(r,n,k,s,h)$ discussed in chapter 4. We make the following general observation. When we require to use the modified algorithm to find the order of a particular group, for example in

3.4.4 we show that $\underset{\sim}{G}(2,2,3)$ is a group of order $2^{15}.3^{3}$, it should be pointed out that although a larger machine than the IBM 360 that we have used for the coset enumeration programme may be able to determine the order directly, the method is nevertheless general and would enable us to sort out finite groups of greater order that the larger machines themselves could not handle directly. (As yet no machine has been constructed that can handle the order of $\underset{\sim}{G}(2,2,3)$ directly.) As well as determining the orders of groups or indices of subgroups, perhaps the more important use we make of the modified algorithm is in theorem proving, see for example 4.6 where we show that the class $\underset{\sim}{H}(r,n,k,s,h)$ contains a subclass of certain metacyclic groups of order $r^n - s^n$ and for these groups, as well as determining the orders, we also find 2-generator 2-relation presentations. We also give in 3.4 and 4.3 examples of finite non-metacyclic 2-generator groups of deficiency zero.

1.3 Preliminary definitions and theorems.

Notation: We use the notation * to denote *free product*, for example $\mathbb{Z}_d \times \mathbb{Z}_d$ denotes the free product of $\mathbb{Z}_d$ with $\mathbb{Z}_d$.

Definition. A *normal series* of a group G is a finite series of subgroups

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \dots \trianglerighteq G_{r-1} \trianglerighteq G_r = 1$$

such that G_i is a normal subgroup of G_{i-1} for $i = 1,2, \dots, r$.

Definition. A *central series* in the group G is a normal series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \dots \trianglerighteq G_{r-1} \trianglerighteq G_r = 1$$

such that

(i) G_i is a normal subgroup of G, for $0 \leq i \leq r$, and

(ii) G_{i-1}/G_i lies in the centre of G/G_i, for $1 \leq i \leq r$.

Definition. A group G is *soluble* if it has a normal series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \dots \trianglerighteq G_{r-1} \trianglerighteq G_r = 1$$

in which G_{i-1}/G_i is abelian for $1 \leq i \leq r$. A group with such a series is said to be soluble of length r.

Definition. The *derived group* (or commutator subgroup) of G is the subgroup generated by all the commutators in G. Denote the derived group by G'.

Definition. A group G is *metacyclic* if it has a normal subgroup N such that G/N and N are cyclic.

Definition. A group G is *metabelian* (or soluble of length two) if and only if G' is abelian.

Definition. A group G is *perfect* if $G = G'$, the commutator subgroup.

Definition. A group G is *nilpotent* if it has a central series.

Theorem 1.3.1. A finite group G is nilpotent if, and only if, it is the direct product of its Sylow p-subgroups.

For a proof see Macdonald (35).

Definition. The *normal closure* of a subgroup H of a group G is the smallest normal subgroup of G which contains H.

Definition. *Deficiency of a presentation.* If G is a finitely presented group, then a finite presentation $\mathcal{P}$ of G is said to have deficiency $m - n$ if it defines G with m generators and n relations.

Definition. The *deficiency* of G is the maximum of the deficiencies of all the finite presentations $\mathcal{P}$ of G.

See (35) for a proof that a group with finite presentation and positive deficiency is necessarily infinite. As a consequence we have the following result.

Theorem 1.3.2. If the group G is finite, the deficiency of G is less than or equal to zero.

Finite metacyclic groups of deficiency zero have been classified by Wamsley (44) and by Beyl (4). Such groups may be presented as follows:

$$\langle a,b \mid a^m = 1,\ b^{-1}ab = a^r,\ b^n = a^{-\frac{m}{(m,r-1)}} \rangle$$

where $r^n \equiv 1 \bmod m$, $m,n \in \mathbb{N}$ and $r \in \mathbb{Z}$.

The class of finite groups of deficiency zero known to have 2-generator 2-relation presentations is small. There is, for example, a class of nilpotent groups given by Macdonald (34), a class of groups given by Wamsley (45), some examples in Coxeter and Moser (24) and, as already mentioned, some examples of our own that we give in chapter 3 and chapter 4. A further recent example is given by Wamsley (46) who shows that, for p an odd prime, the maximal p-factor of the Macdonald groups $G(\alpha,\beta)$, see (34), has zero deficiency. If we insist that in addition the groups are cyclically presented (see definition following Theorem 1.3.3) then, apart from our own examples, even fewer examples are known.

We now give a theorem of D. L. Johnson, J. W. Wamsley and D. Wright (31).

Theorem 1.3.3. If G is a finite group of deficiency zero then G/G' is a 3-generator group.

We now consider what is meant by a group being cyclically presented. Let F_n be the free group on $\{a_i : i \in \mathbb{Z}_n\}$, where the set of congruence classes mod n is used as an index set for the generators. Let ϕ be the permutation $(1\ 2\ \dots\ n)$ of $\mathbb{Z}_n$ and denote by θ the automorphism of F_n induced by ϕ, namely $a_i\theta = a_{i\phi}$. Given a word $w \in F_n$ define the group $G_n(w)$ to be the group F_n/N where N is the smallest θ-invariant normal subgroup of F_n containing w. Then $G_n(w) \cong \langle a_1, a_2, \dots, a_n \mid w\theta^{i-1} = 1,\ 1 \le i \le n \rangle$.

<u>Definition.</u> A group G is said to be *cyclically presented* if $G \cong G_n(w)$ for some $n \in \mathbb{N}$, some $w \in F_n$.

1.4 <u>(l,m,n) groups.</u>

We now give some results about (l,m,n) groups and their generalisations, see for example (24). These results are used in 3.3 and 4.4. The polyhedral groups (l,m,n) $l,m,n \geq 2$ have a presentation

$$\langle R,S,T \mid R^l = S^m = T^n = RST = 1 \rangle,$$

or

$$\langle R,S \mid R^l = S^m = (RS)^n = 1 \rangle.$$

The polyhedral group (l,m,n) is finite if, and only if, the number

$$k = lmn\left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1\right) = mn + nl + lm - lmn$$

is positive, that is in the cases $(2,2,n)$, $(2,3,3)$, $(2,3,4)$ and $(2,3,5)$, the order of (l,m,n) being $2lmn/k$. Thus, for example, $(2,d,d)$ is infinite when $d \ge 4$.

We also require to consider the larger groups $\langle l,m,n \rangle; l,m,n > 1$ defined by

$$\langle R,S,T \mid R^l = S^m = T^n = RST (= Z \text{ say}) \rangle.$$

Since (l,m,n) occurs as a factor group, $\langle l,m,n \rangle$ is infinite when $k \le 0$. When $k > 0$ it can be shown (23) that the period of Z is 2. $\langle l,m,n \rangle$ is called ~~the~~ a binary polyhedral group; its order is twice that of (l,m,n) namely $4lmn/k$.

When $l = 2$ we obtain for $\langle 2,m,n \rangle$ from the presentation for $\langle l,m,n \rangle$

$$\langle R,S,T \mid R^l = S^m = T^n = RST \rangle$$

the presentation

$$\langle S,T \mid S^m = T^n = (ST)^2 \rangle$$

or

$$\langle S,T \mid TST = S^{m-1},\ STS = T^{n-1} \rangle.$$

Thus the binary tetrahedral group $\langle 2,3,3 \rangle$ has a presentation

$$\langle S,T \mid TST = S^2,\ STS = T^2 \rangle.$$

We also have a generalisation to the groups $\langle -l,m,n \rangle; l,m,n > 1$ defined by

$$\langle R,S,T \mid R^{-l} = S^m = T^n = RST \rangle.$$

Writing $u = \frac{2mn}{k} - 1$, it can be shown (22) that

$$\langle -l,m,n \rangle \cong \langle l,m,n \rangle \times \mathbb{Z}_u .$$

Another generalisation is to the groups $\langle l,m \mid n \rangle; l,m,n > 1$, defined by

$$\langle R,S \mid R^l = S^m,\ (RS)^n = 1 \rangle.$$

Its order is that of (l,m,n) multiplied by the period of the central element $Z = R^l = S^m$. We also have the groups $\langle -l,m \mid n \rangle; l,m,n > 1$ defined by

$$\langle R,S \mid R^{-l} = S^m,\ (RS)^n = 1 \rangle,$$

or

$$\langle R,S \mid R^l S^m = 1,\ (RS)^n = 1 \rangle.$$

The group $\langle -l,m \mid n \rangle$ is clearly infinite when $l = m$, since the extra relation $RS = 1$ gives the infinite cyclic group as a factor group. Another presentation for $\langle -l,m \mid n \rangle$ is

$$\langle S,T \mid S^m = (ST)^l,\ T^n = 1 \rangle.$$

It is shown in (24) that $\langle -2,3 \mid m \rangle$ has a presentation

$\langle S,T \mid S^m = 1,\ STS = TST \rangle$.

It is also shown that ~~$\langle -1,m|n \rangle$ has order $\frac{4(m-1)lmn^2}{k^2}$ where~~
~~$k = mn + nl + lm - lmn$ divides (l,m) and is positive.~~ Thus $\langle -2,3|n \rangle$
has order $\frac{4.6n^2}{(6-n)^2} = \frac{24n^2}{(6-n)^2}$ if $2 \leq n \leq 5$.

Definition. A finite presentation of a group is a group presentation consisting of a finite free basis and a finite set of words.

Definition. A finitely presented group is a group which has a finite presentation.

CHAPTER 2

THE TODD-COXETER COSET ENUMERATION ALGORITHM

2.1 Introduction.

In this chapter we describe the Todd-Coxeter coset enumeration algorithm and the modified algorithm and show how, given a finitely generated subgroup H ~~of finite index~~ in a finitely generated finitely presented group G, we can find a presentation for H. We then give elementary examples of the use of the algorithms, the examples being special cases of results in chapter 4. The chapter concludes with a discussion on the computer programmes that have been used in connection with the material in chapter 3 and chapter 4.

2.2 The Todd-Coxeter coset enumeration algorithm and the modified algorithm.

In this section we describe the Todd-Coxeter coset enumeration algorithm giving it essentially in the form in which it has been implemented on the IBM 360/44 computer of the University of St. Andrews Computing Laboratory. We then describe the modified algorithm which, given a finitely generated subgroup H of finite index in a finitely generated finitely presented group G, will give a finite presentation for H. I have already given a description of the modified algorithm, see (7), and such a modification has also been discussed in joint work by Beetham and Campbell (2) and by several other authors (3), (19), (33), (38). A theorem about the presentation for the subgroup and results about the algorithms and the proofs of the validity of their operation were written up in (2). We now give them in sections 2.2 and 2.3, for these results are crucial for the applications in chapter 3 and chapter 4. The proofs become much simplified when looked at in the context of free groups, this idea being due to M. J. Beetham, while the writing of the paper was to a considerable extent my work. The other point of note is that our modification to the algorithm enables us to find a presentation for a finitely generated subgroup of countable index in a ~~finitely generated~~ finitely presented group by using just one enumeration, other authors handling only finite index and that with two enumerations.

Suppose a ~~group G~~ free group F is generated by a set of elements $\{x_i | i \in I\}$, I a finite set and suppose that $\{\sigma_k | k \in K\}$, K a finite set, is a set of words in the generators x_i and their inverses, then we will use the notation $G = \langle x_i | \sigma_k = 1,\ i \in I,\ k \in K \rangle$ to mean the group $F/\overline{L}$, where $\overline{L}$ is the normal closure of the group $L = \langle \sigma_k | k \in K \rangle$ in the free group $F = \langle x_i | i \in I \rangle$.

If $H = \langle w_j | j \in J \rangle$, J a finite set, is a subgroup of ~~countable index in~~ G there are only a countable number of cosets of H in G. We first give a systematic method for finding these cosets. (This in fact extends the original algorithm (42) which was concerned with enumerating the cosets of a finitely generated subgroup of finite index in a finitely presented group.) We construct a multiplication table for the cosets in which the rows are indexed by the cosets and the columns by the generators of G and their inverses and proceed in the following way:

(1) Denote by the integer 1 the subgroup H. Apply the subgroup generators w_j to the coset 1, defining new cosets as necessary to obtain $1.w_j = 1$ for all $j \in J$. (The algorithm thus described is as implemented on the computer (1). Other forms of the algorithm may also be used, see section 2.4.)

(2) Next, for each coset m in turn in the sequence 1,2,3, ...,

(a) define the cosets mx_i and mx_i^{-1} for each $i \in I$ unless these have already been defined or found,

(b) apply the relations to the coset m, defining any new cosets which are necessary, to obtain $m\sigma_k = m$ for all $k \in K$.

Note that any new coset defined in (1) and (2) must be defined as a previously defined coset multiplied either by a generator x_i or the inverse of a generator. Also the relation $mx_i^{\varepsilon} = n$ implies $nx_i^{-\varepsilon} = m$, where $\varepsilon \in \{-1,1\}$.

(3) In addition a collapse (sometimes called coincidence) may occur. Suppose a relation is $x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_i^{\varepsilon_i} = 1$ $(1,2,\cdots,i \in I)$ and we apply that relation to the coset n, then collapse occurs if we obtain $nx_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_j^{\varepsilon_j} = p$, where $1 \le j \le i$, and we also have $nx_i^{-\varepsilon_i} x_{i-1}^{-\varepsilon_{i-1}} \ldots x_{j+1}^{-\varepsilon_{j+1}} = q$, that is $nx_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_j^{\varepsilon_j} = q$ for the labels p and q are both attached to the same coset.

Having discovered a collapse ($q > p$) the *pth* and *qth* rows of the multiplication table are compared, possibly giving further collapses, and then q is replaced everywhere it occurs by p. These further collapses are processed in the same manner. At the end of the algorithm the cosets can be relabelled by the first available integers.

Suppose the subgroup generator $w_j = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_i^{\varepsilon_i}$ and we consider $1.w_j$, then collapse may again occur, for we may discover that $1.x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_k^{\varepsilon_k} = p$, where $1 \le k \le i$ whereas $1.x_i^{-\varepsilon_i} x_{i-1}^{-\varepsilon_{i-1}} \ldots x_{k+1}^{-\varepsilon_{k+1}} = q$. Thus p and q are both attached to the same coset and we then process the resulting collapses as above.

Finally, the label chosen for an unknown coset should be the first available natural number. This completes the coset enumeration algorithm.

We next give the modified algorithm:

(4) Define the identity element of the subgroup H to be the coset representative of the coset 1. Whenever a coset n is defined as mx_i^{ε} define the coset representative τ_n to be $\tau_m x_i^{\varepsilon}$, where τ_m is the coset representative of the coset m, and note that $\tau_m x_i^{\varepsilon} \tau_n^{-1} = 1$.

(5) If, when considering a generator w of H of the form sxt, where s and t are words in the generators of G and x is either a generator or the inverse of a generator of G, we find that mx = n, where m is the coset 1s and n is the coset $1t^{-1}$, then note that $\tau_m x \tau_n^{-1} = \alpha^{-1} w \beta^{-1}$, where α is a word in the generators of H calculated as $1x_1^{\varepsilon_1}\tau_{i_1}^{-1}\tau_{i_1}x_2^{\varepsilon_2} \dots \tau_{i_{k-1}} x_k^{\varepsilon_k}\tau_m^{-1}$ where $s = x_1^{\varepsilon_1}x_2^{\varepsilon_2} \dots x_k^{\varepsilon_k}$ and where β is calculated in a similar way. τ_{i_j} is the representative of the coset containing $\tau_{i_{j-1}} x_j^{\varepsilon_j}$.

(6) If, when considering a relation of G of the form sxt = 1, we find that mx = n, then note that $\tau_m x \tau_n^{-1} = \alpha^{-1}\chi\beta^{-1}$ where α and β are words in the generators of H calculated as in (5).

(7) If, when considering a generator w of H, no new cosets are defined and no new information about coset multiplication is found then $\prod_{j=0}^{r}\left(\tau_{i_j} x_{j+1}^{\varepsilon_{j+1}} \tau_{i_{j+1}}^{-1}\right) = w$ is a relation between the generators of H, where $w = \prod_{j=0}^{r} x_{j+1}^{\varepsilon_{j+1}}$.

(8) If, when considering a relation σ applied to a coset m, no new cosets are defined and no new information about coset multiplication is found then $\prod_{j=0}^{r}\left(\tau_{i_j} x_{j+1}^{\varepsilon_{j+1}} \tau_{i_{j+1}}^{-1}\right) = 1$ is a relation between the generators of H, where $\sigma = \prod_{j=0}^{r} x_{i_{j+1}}^{\varepsilon_{j+1}}$ and $\tau_{i_0} = \tau_{i_{r+1}}$ is the coset representative of m.

(9) The last occurrence to be considered is the discovery of a collapse. If, when considering a generator of the subgroup, w_1 say, it is found that w_1 is of the form st where Hs = m and $Ht^{-1} = n$, then this gives $\tau_1 s = w_s \tau_m$ and $\tau_1 t^{-1} = w_t \tau_n$ from which $\tau_n = (w_t^{-1} w_1^{-1} w_s)\tau_m$. This gives the relationship between the coset representatives and again only involves words in the generators of H.

If, when considering a relation of G, a collapse occurs then a similar result is obtained. A relation of the form $\tau_n g = w\tau_l$ becomes $\tau_m g = w_s^{-1} w_1 w_t w \tau_l$. If the result mg = l is already known then this gives a further relation in H. Otherwise this is new information resulting from the collapse. If it is already known that mg = k with $\tau_m g = w'\tau_k$ so that the collapse l = k is obtained then $\tau_l = w^{-1} w_t^{-1} w_1^{-1} w_s w' \tau_k$ so that the l's can be replaced by k's and the τ_l's by this expression involving τ_k and a word in the generators of H.

2.3 Proofs of results.

In this section we give proofs of the results contained in the Todd-Coxeter coset enumeration algorithm and the modified Todd-Coxeter coset enumeration algorithm as described in section 2.2.

In Lemma 2.3.1 we are concerned with a finitely generated subgroup H of countable index in a free group F, in Lemma 2.3.2 with a finitely generated subgroup H of countable index in a finitely generated finitely presented group G, and in Theorem 2.3.3 with a finitely generated subgroup H of finite index in a finitely generated finitely presented group G. In Theorem 2.3.4 we show how to find a presentation for a finitely generated subgroup H of countable index in a finitely generated finitely presented group G and show that when the index is finite then the presentation for H is finite.

<u>Lemma 2.3.1</u> Let F be a free group and H a finitely generated subgroup $\langle w_j \mid j \in J \rangle$ of countable index in F. If $\{x_i \mid i \in I\}$ is a set of generators of F, then the identities $Hw_j = H$ and $H\theta x_i^{\varepsilon} x_i^{-\varepsilon} = H\theta$, $\varepsilon = \pm 1$, for all cosets $H\theta$, are sufficient to enumerate the cosets of H in F.

<u>Proof</u> Suppose that the coset $H\mu = H\nu$, where μ and ν are words in the generators of F. Then $\mu\nu^{-1} \in H$, that is $\mu\nu^{-1} = w_{j_1}^{\varepsilon_1} w_{j_2}^{\varepsilon_2} \ldots w_{j_n}^{\varepsilon_n}$ for some $\{j_1, j_2, \ldots, j_n\} \subseteq J$ and for some $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\} \subseteq \{-1, 1\}$. Note that because F is free this equality is an identity apart from words of the form $\eta\eta^{-1}$ on either side of the equation.

~~Now for each sequence~~ $\cancel{x_{i_1}^{\varepsilon_1}, x_{i_2}^{\varepsilon_2}, \ldots, x_{i_n}^{\varepsilon_n}}$ ~~define the coset~~ $\cancel{H_{i_1, i_2, \ldots, i_n}^{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n}}$ ~~to be the coset~~ $\cancel{Hx_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \ldots x_{i_n}^{\varepsilon_n}}$. If we now calculate $H\mu\nu^{-1}$ we find that this is H provided $H\theta x_i^{\varepsilon} x_i^{-\varepsilon} = H\theta$ for all cosets $H\theta$ and all generators x_i and provided $Hw_j = H$ for all $j \in J$ (this implies $Hw_j^{-1} = H$). Thus to find all the equalities between the cosets it is sufficient to check that $H\theta x_i^{\varepsilon} x_i^{-\varepsilon} = H\theta$ and that $Hw_j = H$ for all the generators w_j of H.

In examples where we are considering a finite number of cosets, n say, then it is more convenient to label the cosets 1,2, ..., n rather than use the notation described above.

If we next ask what is the index of the group generated by a set of words $\{w_j | j \in J\}$ in the group $G = \langle x_i | i \in I, \sigma_k = 1, k \in K \rangle$, then the corresponding question about free groups is that of finding the index of the subgroup generated by the set $\{w_j | j \in J\}$ and the normal closure of the group $\langle \sigma_k | k \in K \rangle$ in the free group $\langle x_i | i \in I \rangle$. The subgroup is not necessarily given as a finitely generated group since the only set of generators we have is $\{w_j | j \in J\} \cup \{w^{-1}\sigma_k w | k \in K, w \in \langle x_i | i \in I \rangle\}$. However if G is finitely generated and finitely presented, that is the sets I and K are both finite, and the subgroup H is ~~of countable index and~~ finitely generated, then it is sufficient to consider a countable number of relations of the type given in Lemma 2.3.1.

Lemma 2.3.2. Let H be a finitely generated subgroup $\langle w_j | j \in J \rangle$ ~~of countable index~~ in a finitely generated finitely presented group $G = \langle x_i | \sigma_k = 1, i \in I, k \in K \rangle$ then the identities $Hw_j = H$, $H\theta\sigma_k = H\theta$ and $H\theta x_i^{\varepsilon} x_i^{-\varepsilon} = H\theta$, $\varepsilon = \pm 1$ are sufficient to enumerate all the cosets of H in G.

Proof. A typical generator $w^{-1}\sigma_k w$ is equivalent to the relation $H\theta\sigma_k = H\theta$, where $H\theta$ is the coset $1w^{-1}$. Thus we have reduced the problem to investigating, for each coset $H\theta$, the relation $H\theta\sigma_k = H\theta$, for each of the given relations $\sigma_k = 1$ of G. Therefore, on adding the identities $H\theta\sigma_k = H\theta$ to the identities in Lemma 2.3.1, we are now able to enumerate all the cosets of H in G.

So far we have only shown that given a finitely generated finitely presented group we can find the index of a finitely generated subgroup in a countable number of steps. We next show that when the index of the subgroup is finite, then a finite number of steps is sufficient.

Theorem 2.3.3. Let H be a finitely generated subgroup $\langle w_j | j \in J \rangle$ of finite index in a finitely generated finitely presented group $G = \langle x_i | \sigma_k = 1,\ i \in I,\ k \in K \rangle$ then the classical algorithm terminates in a finite number of steps.

Proof. If $H\theta$ is any coset of H, then we can choose a representative of $H\theta$, g say. Expressing g as a word in the generators of G we have $H\theta = 1.g$ and step (2a) of the algorithm ensures that $H\theta$ is given at least one label in the above process in a finite number of steps. As H is of finite index in G we can then find an integer m such that each coset has a label less than m.

We must now show that, again in a finite number of steps, we can identify the cosets ng and ng^{-1} among those m for all non-redundant $n \leq m$ and all g in the generating set. As the number of such cosets is finite we need only show that ng can be identified in a finite number of steps, where g is either a generator or the inverse of a generator. By the choice of m, ng is equal to some coset k say, with $k \leq m$. n is defined as 1σ for some σ in G and similarly k is defined as 1w for some w in G.

Now ng = k is equivalent to $\sigma g w^{-1} \in H$. However $\sigma g w^{-1} \in H$ if and only if it can be written as a product $\prod \mu_i$, where μ_i is either a generator of H, the inverse of a generator of H, a conjugate of a relation of G or the inverse of such a conjugate.

But after a finite number of steps the algorithm provides sufficient information to give $1\mu_i = 1$ and so $1\sigma g w^{-1} = 1$. Thus all the cosets ng can be identified in a finite number of steps and so the process must terminate for it is then possible to complete the coset enumeration without defining further cosets.

Notice that we cannot restrict our attention only to the first m cosets. In fact there is no upper bound to the number of cosets needed to complete the process in terms of the index, the lengths of the generators and relations and the number of generators and relations.

Finally in this section we consider the presentation for H obtained from the modified Todd-Coxeter coset enumeration algorithm where H is a finitely generated subgroup in a finitely generated finitely presented group.

<u>Theorem 2.3.4.</u> Let H be a finitely generated subgroup $\langle h_j \mid j \in J \rangle$ ~~of countable index~~ in a finitely generated finitely presented group $G = \langle x_i \mid \sigma_k = 1,\ i \in I,\ k \in K \rangle$. Then the modified algorithm provides a presentation for H. Further, if the index of H in G is finite, this presentation is finite.

<u>Proof.</u> Suppose $h_{i_1}^{\varepsilon_1} h_{i_2}^{\varepsilon_2} \dots h_{i_n}^{\varepsilon_n} = 1$ is a relation between the generators of H. Then, as this is a consequence of the relations of G, $h_{i_1}^{\varepsilon_1} h_{i_2}^{\varepsilon_2} \dots h_{i_n}^{\varepsilon_n}$ is freely equal to $\prod \mu_j$, where μ_j is either a conjugate of a relation of G or the inverse of a conjugate.

$1\mu_j = 1$ modulo the relations of H discovered during the process, that is $1\mu_j = 1$ or $1\mu_j = w.1$ where $w = 1$ is one of the relations.

Therefore $1.\prod \mu_j = 1$ modulo the relations of H discovered during the coset enumeration. On the other hand the initial consideration of generators of H shows that $1.h_i = h_i.1$ modulo any later changes brought about by using relations of H discovered in consequence of collapses occurring during the algorithm. Thus, as required, the relation $h_{i_1}^{\varepsilon_1} h_{i_2}^{\varepsilon_2} \ldots h_{i_n}^{\varepsilon_n} = 1$ is a consequence of the relations discovered by the algorithm.

2.4 Examples.

It is worth noting that the examples in this section are special cases of the main work of chapter 4 and therefore repay following in detail.

As a first example of the Todd-Coxeter coset enumeration algorithm let us consider the group G with presentation

$$\langle a,b,c \mid abcab^{-1} = bcabc^{-1} = cabca^{-1} = 1 \rangle.$$

Suppose we wish to find the index in G of the subgroup H generated by abc. Denote the coset H by 1. Our initial information is that Habc = H so we have 1abc = 1. We record the information in tabular form. Let each of the given relations serve as the heading for a table of cosets and if the relation is of length l then we will have a table of cosets with l+1 columns. In the same way we write down the subgroup generators. We also have a table showing how to multiply each coset by each generator and inverse of a generator. For the above example the tables are initially as follows:

$abcab^{-1}$		$bcabc^{-1}$		$cabca^{-1}$		abc	
1	1	1	1	1	1	1	1

Suppose we define coset 2 by $2 = 1.a$ and coset 3 by $3 = 2.b$. Then, since $1abc = 1$ we have $3c = 1$. Now use the fact that $1abcab^{-1} = 1$. We know that $1a = 2$, $2b = 3$, $3c = 1$, $1a = 2$ so $1abca = 2$. Hence $2b^{-1} = 1$ giving the new information $1b = 2$. In the tables we underline the position at which we have found new information. The tables now become

	a	b	c	a	b^{-1}
1	2	3	1	2	1

	b	c	a	b	c^{-1}
1					1

	c	a	b	c	a^{-1}
1					1

	a	b	c
1	2	3	1

	a	b	c	a^{-1}	b^{-1}	c^{-1}
1	2	2				3
2		3		1	1	
3			1		2	

We continue to fill in the tables giving:

	a	b	c	a	b^{-1}
1	2	3	1	2	1

	b	c	a	b	c^{-1}
1	2				1

	c	a	b	c	a^{-1}
1				2	1

	a	b	c
1	2	3	1

Now we also have $2abcab^{-1} = 2$, $2bcabc^{-1} = 2$ and $2cabca^{-1} = 2$ so the tables become

	a	b	c	a	b^{-1}
1	2	3	1	2	1
2				3	2

	b	c	a	b	c^{-1}
1	2				1
2	3	1	2	3	2

	c	a	b	c	a^{-1}
1				2	1
2					2

	a	b	c
1	2	3	1

giving the new information $2c = 3$. Also $3abcab^{-1} = 3$, $3bcabc^{-1} = 3$, $3cabca^{-1} = 3$ and so we now have

	a	b	c	a	b^{-1}
1	2	3	1	2	1
2				3	2
3					3

	b	c	a	b	c^{-1}
1	2				1
2	3	1	2	3	2
3					3

	c	a	b	c	a^{-1}
1				2	1
2					2
3	1	2	3	1	3

	a	b	c
1	2	3	1

	a	b	c	a^{-1}	b^{-1}	c^{-1}
1	2	2		3		3
2		3	3	1	1	
3	1		1		2	2

giving 3a = 1. The tables now complete giving 2a = 3, 3b = 1 and 1c = 2.

	a	b	c	a	b^{-1}
1	2	3	1	2	1
2	3	1	2	3	2
3	1	2	3	1	3

	b	c	a	b	c^{-1}
1	2	3	1	2	1
2	3	1	2	3	2
3	1	2	3	1	3

	c	a	b	c	a^{-1}
1	2	3	1	2	1
2	3	1	2	3	2
3	1	2	3	1	3

	a	b	c
1	2	3	1

	a	b	c	a^{-1}	b^{-1}	c^{-1}
1	2	2	2	3	3	3
2	3	3	3	1	1	1
3	1	1	1	2	2	2

Since 1a = 2, 2a = 3, 3a = 1, 1b = 2, 2b = 3, 3b = 1, 1c = 2, 2c = 3 and 3c = 1 the process is complete and all the cosets of H have been found. Thus H has index 3 in G.

Let us now consider the modified coset enumeration algorithm. We take the same example as before and look for a presentation for H. Let h = abc so that H = ⟨h⟩. We now consider coset representatives rather than cosets. Take as the coset representative of H the element 1. Then 2 = 1 a and 3 = 2b. From the subgroup generator abc, 3c = h1. This time we have a table giving the relations between the coset representatives:

	a	b	c	a	b^{-1}
1	2				1

	b	c	a	b	c^{-1}
1					1

	c	a	b	c	a^{-1}
1					1

	a	b	c
1	2	3	1

	a	b	c	a^{-1}	b^{-1}	c^{-1}
1	2					$h^{-1}3$
2		3		1		
3			h 1		2	

From the relation $1abcab^{-1} = 1$, $1abca = h2$ and so $h2b^{-1} = 1$, that is $1b = h2$. Similarly from $2bcabc^{-1} = 2$, $2bcab = 3cab = h1ab = h2b = h3$ and so $h3c^{-1} = 2$, that is $2c = h3$. From $3cabca^{-1} = 3$, $3cabc = h1abc = h2bc = h3c = h^2 1$ and thus $3a = h^2 1$. The tables now become

a	b	c	a	b^{-1}	
1	2	3	1	2	1
2				3	2
3					3

b	c	a	b	c^{-1}	
1	2				1
2	3	1	2	3	2
3					3

c	a	b	c	a^{-1}	
1				2	1
2					2
3	1	2	3	1	3

a	b	c	
1	2	3	1

	a	b	c	a^{-1}	b^{-1}	c^{-1}
1	2	$h2$		$h^{-2}3$		$h^{-1}3$
2		3	$h3$	1	$h^{-1}1$	
3	$h^2 1$		$h1$		2	$h^{-1}2$

As before the tables complete, this time giving $2a = h^5 3$, $3b = h^6 1$ and $1c = h^5 2$.

a	b	c	a	b^{-1}	
1	2	3	1	2	1
2	3	1	2	3	2
3	1	2	3	1	3

b	c	a	b	c^{-1}	
1	2	3	1	2	1
2	3	1	2	3	2
3	1	2	3	1	3

c	a	b	c	a^{-1}	
1	2	3	1	2	1
2	3	1	2	3	2
3	1	2	3	1	3

a	b	c	
1	2	3	1

	a	b	c	a^{-1}	b^{-1}	c^{-1}
1	2	$h2$	$h^5 2$	$h^{-2}3$	$h^{-6}3$	$h^{-1}3$
2	$h^5 3$	3	$h3$	1	$h^{-1}1$	$h^{-5}1$
3	$h^2 1$	$h^6 1$	$h1$	$h^{-5}2$	2	$h^{-1}2$

The relations for H are found from $2abcab^{-1} = 2$, $3bcabc^{-1} = 3$ and $1cabca^{-1} = 1$, the other rows in the tables giving only the trivial relation $1 = 1$. For example, from $1bcabc^{-1} = 1$,

$$1bcabc^{-1} = h2cabc^{-1} = h^2 3abc^{-1} = h^4 1bc^{-1} = h^5 2c^{-1} = h^5 h^{-5} 1 = 1.$$

From the relation $2abcab^{-1} = 2$,

$$2abcab^{-1} = h^5 3bcab^{-1} = h^5 h^6 1cab^{-1} = h^{11} h^5 2ab^{-1} = h^{16} h^5 3b^{-1} = h^{21} 2.$$

Therefore $h^{21} = 1$. Similarly from each of $3bcabc^{-1} = 3$ and $1cabca^{-1} = 1$ we also get $h^{21} = 1$. Thus H has a presentation $H \cong \langle h \mid h^{21} = 1 \rangle$.

Suppose we now consider an example of the coset enumeration algorithm where collapse occurs. Again take the group $G \cong \langle a,b,c \mid abcab^{-1} = bcabc^{-1} = cabca^{-1} = 1 \rangle$ but this time consider the subgroup $H = \langle ab, abc \rangle$. We know that $Hab = H$ and $Habc = H$, so that if we denote the coset H by 1 then $1ab = 1$ and $1abc = 1$. Suppose we define coset 2 by $2 = 1a$ then since $1ab = 1$, $2b = 1$. Further, since $1abc = 1$, $1c = 1$. Now use the fact that $1abcab^{-1} = 1$. We know that $1a = 2$, $2b = 1$, $1c = 1$ and $1a = 2$, so $1abca = 2$. Hence $2b^{-1} = 1$ giving the new information $1b = 2$.

The tables become

	a	b	c	a	b^{-1}
1	2	1	1	$\underline{2}$	1
2					2

	b	c	a	b	c^{-1}
1					1
2	1	1	2	$\underline{1}$	2

	c	a	b	c	a^{-1}
1					1
2					2

	a	b
1	2	1

	a	b	c
1	2	1	1

	a	b	c	a^{-1}	b^{-1}	c^{-1}
1	2		1		2	1
2		1		1		

and we have obtained the new information $2c = 1$. However $1c = 1$ so that coset 1 = coset 2. Thus we have $1a = 1$, $1b = 1$ and $1c = 1$, giving complete collapse.

Now look at the same example and use the modified algorithm. Let $h_1 = ab$ and $h_2 = abc$. Then we have the partial tables

	a	b	c	a^{-1}	b^{-1}	c^{-1}
1	2	$h_2 2$	$h_1^{-1}h_2 1$		$h_1^{-1}2$	$h_2^{-1}h_1 1$
2		$h_1 1$		1	$h_2^{-1}1$	

From the relation $2bcabc^{-1} = 2$

$$2bcab = h_1 1cab = h_2 1ab = h_2 2b = h_2 h_1 1$$

and so $2c = h_2 h_1 1$. However $1c = h_1^{-1} h_2 1$ and so we have complete collapse giving $2 = h_2 h_1 h_2^{-1} h_1 1$. Therefore $1a = h_2 h_1 h_2^{-1} h_1 1$, $1b = h_1^{-1} h_2 h_1^{-1} h_2^{-1} h_1 1$ and $1c = h_1^{-1} h_2 1$.

The subgroup generators ab and abc and the relation $bcabc^{-1} = 1$ give no non-trivial relations for H. From $1abcab^{-1} = 1$ we get

$$\begin{aligned}
1abcab^{-1} &= h_2 h_1 h_2^{-1} h_1 1bcab^{-1},\\
&= h_2 h_1 h_2^{-1} h_1 h_1^{-1} h_2 h_1^{-1} h_2^{-1} h_1 1cab^{-1},\\
&= h_1 h_1^{-1} h_2 1ab^{-1},\\
&= h_2 h_2 h_1 h_2^{-1} h_1 1b^{-1},\\
&= h_2^2 h_1 h_2^{-1} h_1 h_1^{-1} h_2 h_1 h_2^{-1} h_1 1,\\
&= 1,
\end{aligned}$$

that is $h_2^2 h_1^2 h_2^{-1} h_1 = 1$.

From $1cabca^{-1} = 1$ we get

$$h_2^2 h_1^{-1} h_2 h_1^{-1} h_2^{-1} h_1^{-1} = 1.$$

Thus $H \cong \langle h_1, h_2 \mid h_1 h_2^2 h_1^2 = h_2,\ h_2^2 h_1^{-1} h_2 = h_1 h_2 h_1 \rangle$, and since the index of H in G is one, this is an alternative presentation for G.

Note that in both cases considered the modified algorithm shows that G is metacyclic. In the first case when we consider the subgroup H generated by abc we show that H is a normal subgroup of index 3 and since abc has order 21, G is metacyclic of order 63 being an extension of the cyclic group $\mathbb{Z}_{21}$ by $\mathbb{Z}_3$. We obtain similar information from the second example where we consider the subgroup generated by $h_1 = ab$ and $h_2 = abc$ but with the new presentation for G obtained in this example, we can show that G has a presentation

$$\langle h_1, h_2 \mid h_1 h_2 h_1^{-1} = h_2^4,\ h_1^3 = h_2^{14},\ h_2^{21} = 1 \rangle$$

and thus we have a presentation for G in standard form.

2.5 Computer programmes.

In this section we give some details of the computer programmes used in our work. We have used two coset enumeration programmes.

The first is one by M. J. Beetham (1). As mentioned earlier, this programme has been implemented on the IBM 360/44 computer of the University of St. Andrews Computing Laboratory and is a much developed version of a programme that I wrote for an IBM 1620 computer. Given a group G and a subgroup $H = \langle h_1, h_2, \ldots, h_n \rangle$ then the programme is able to determine an index of H in G of up to 32,000/(2n+1) where n is the number of subgroup generators. If we take n = 2 then the theoretical maximum index attainable is 6400. In practice we have been able to find indices of just over 5000. This compares with a practical figure of around 20 for hand calculations, hand calculations having the severe limitations of time and effort. As regards the computer time involved then the programme (1) is able to determine in just under two and a half minutes that the order of the group

$$\underset{\sim}{F}(3,6) \cong \langle a,b,c,d,e,f \mid abc = d,\ bcd = e,\ cde = f,\ def = a,\ efa = b,\ fab = c \rangle$$

is 1512.

Note too that in most examples more cosets than the actual index of the subgroup require to be defined during the process, and then of course, as described in 2.2, collapse occurs. As well as printing out the index the programme also prints out the number of collapses involved and this gives in certain cases an indication as to whether an attempt to use the modified Todd-Coxeter algorithm will be successful.

The modified algorithm can be used either for hand calculation which soon becomes hard or for machine calculation. However, the disadvantage of the algorithm is that the number of relations obtained by the algorithm for the subgroup is approximately the number of relations in the presentation of the original group times the index of the subgroup in the group. Also, the length of the relations rapidly increases with the index.

Unfortunately the modified algorithm is not as yet part of the programme (1). We have therefore used two programmes to find a presentation for a finitely generated subgroup of finite index in a finitely generated finitely presented group. We use first a programme by N. W. G. Wilde (47), an IBM programme package entitled the 'Benson Mendelsohn algorithm for certain word problems in groups', the programme implementing the algorithm developed in (3). The programme is in two parts, the first part being a coset enumeration programme and the second extending the algorithm as detailed in the programme specification which now follows.

Let G be a group given by up to 128 generators and up to 32 relations. Let H be a subgroup of G generated by up to 8 words in the group generators. For each coset i and each group generator x_j the programme finds the following relationships involving cosets and a set of coset representatives:

$$ix_j = k,$$

$$\tau_i x_j = w_{ij}\tau_k,$$

where k is a coset, τ_i and τ_k are coset representatives for cosets i and k respectively and w_{ij} is a word in the subgroup generators.

The programme has been implemented for the IBM 360 in assembler language. However, it uses a disk operating system and this is very slow and inefficient on the St. Andrews computer. The programme for the coset enumeration is logically similar to that of Trotter (43). The second part of the programme determines the words w_{ij} in the relationships $\tau_i x_j = w_{ij}\tau_k$. However it does not give a presentation for the subgroup H and so I had to write a programme CCRG to complete the modified algorithm.

We use the output of the programme (47). It is theoretically possible to obtain this output in the form of punched cards but the system compatibility will only allow printed output for the St. Andrews IBM 360. We therefore type input cards as follows.

On card 1, we type a right bracket in position 1.

Card 2 has format 4X,I5,13X,I5,23X,I5,25X where 'X' denotes blanks, 'I' denotes integer format and the three entries I5 are NCOSET the number of cosets as determined by (47), NGGEN the number of generators of G and NGEN the number of subgroup generators.

Card 3 has format 4I3, the entries being the coset 1, the generator x_1, the coset k and the length of the word w_{11}, where length is measured as follows: if $w_{11} = h_{\alpha_1}^{\beta_1} h_{\alpha_2}^{\beta_2} \dots h_{\alpha_n}^{\beta_n}$ where

$h_{\alpha_1}, h_{\alpha_2}, \ldots, h_{\alpha_n}$ are subgroup generators and $\beta_1, \beta_2, \ldots, \beta_n \in \mathbb{Z}\setminus\{0\}$ then the length of $w_{11} = n$.

Card 4, and where necessary additional cards, consists of 12 pairs of numbers I3,I3 where the first number is the label of the subgroup generator concerned and the second is the power to which that generator is raised.

In general we have pairs of cards with the formats of card 3 and card 4, having one pair of cards for each coset acting on each group generator. We consider first coset 1 acting on each group generator, then coset 2 acting on each group generator and so on in general for coset i acting on the group generators $x_1, x_2, \ldots, x_\alpha$ where α is the number of group generators.

The next card has format 4I4, the entries being NGGEN, 0, NGEN and NREL, the number of relations. Then for each relation of the group G we have a pair of cards where the first card in the pair has format I4 and gives the length of the relation, this time the length being defined as follows: if the relation R_j is given by $x_{j_1}^{\beta_1} x_{j_2}^{\beta_2} \ldots x_{j_n}^{\beta_n} = 1$ where $x_{j_1}, x_{j_2}, \ldots, x_{j_n}$ are group generators and $\beta_1, \beta_2, \ldots, \beta_n \in \mathbb{Z}\setminus\{0\}$ then the length of R_j is equal to $|\beta_1| + |\beta_2| + \ldots + |\beta_n|$. The second card, and if necessary further cards, give the relation in the format specified for the Wilde programme (47). Each card has format 19I4 and a relation is entered as a string of group generators and inverses without using exponents. For example if the relation is $x_2^3 x_1 x_2^{-1} = 1$, the relation must be written as

$$x_2 x_2 x_2 x_1 x_2^{-1} = 1$$

and the input card will then be

2 2 2 1 -2

the integers being the subscripts of the group generators. A minus sign indicates the inverse of a generator. Each relation may be continued for as many cards as required with 19 four digit fields to a card.

Now consider the output of the programme CCRG. The line-printer prints out the input cards and also gives the inverses of the words w_{ij}. It also gives, within brackets, the relations for the subgroup or states that the relation found is trivial. The punch provides card output for the relations of the subgroup in a form that can be immediately used by the programme (1). The advantage of having the relations printed as well as punched is that redundant relations may be discarded and relations for the subgroup may also be simplified. Thus in certain applications, see for example chapter 3, the presentation for the subgroup is simplified before implementation of the programme (1).

We have also written another program CCRGAB, a modification of CCRG, in which the relations for the subgroup are abelianised. Such a programme is useful if we know that the subgroup is abelian and an application is given in section 3.4. The input is the same as for CCRG and the output format is as for CCRG but the relations before being printed and punched are abelianised. The programme as at present written will handle up to five subgroup generators h_1, h_2, h_3, h_4, h_5 and will give relations of the form $h_1^{\alpha_1} h_2^{\alpha_2} h_3^{\alpha_3} h_4^{\alpha_4} h_5^{\alpha_5} = 1$.

We give details of the programmes CCRG and CCRGAB in appendix 1.

2.6 Computational methods.

We conclude this chapter by looking at a result that we have used in conjunction with the coset enumeration programme (1).

Cyclic subgroups. Suppose we have a group G and that coset enumeration enables us to determine that a word w_1 in the generators of G and their inverses determines a cyclic subgroup of index n. Further suppose that w_2 is another word in the generators of G and their inverses and that the index of $\langle w_1,w_2\rangle$ in G is still n, then we know that w_2 belongs to the cyclic subgroup generated by w_1. Thus we may add to the presentation for G the relation that w_1 and w_2 commute, and similarly for any further words $w_3,w_4, \ldots, w_k$ that leave the index unaltered. The addition of such relations may, in certain cases, enable the coset enumeration programme (1) to determine the order of the group directly where, without these additional relations, this had previously proved impossible.

Other computational techniques that we have used will be discussed when they are first mentioned. However, when we state that the order of a group has been determined directly by use of the Todd-Coxeter algorithm, then, although not explicitly stated, we may have used this result to determine relations that hold in the group and which simplify the coset enumeration, or, of course, we may have found such relations algebraically. This result about cyclic subgroups is used, for example, in section 3.4.

CHAPTER 3

A CLASS OF GROUPS $\underset{\sim}{G}(l,m,n)$

3.1 Introduction.

In (7) one of the problems I considered was to show how the modified Todd-Coxeter coset enumeration algorithm, see section 2.2, could be used to prove that the group with presentation

$$\langle R,S \mid RS^2 = S^3R,\ SR^2 = R^3S \rangle$$

is in fact the trivial group. (This presentation arises out of a problem in knot theory; see R. H. Fox, 1956, Free differential calculus.)

C. T. Benson and N. S. Mendelsohn (3) were then able to show that the generalisation to the presentation

$$\langle R,S \mid RS^n = S^{n+1}R,\ SR^n = R^{n+1}S \rangle \qquad (\dagger)$$

still gave the trivial group. In (8) I modified the original presentation and considered the class of groups with presentation

$$\langle R,S \mid RS^m = S^{m-1}RSR,\ SR^m = R^{m-1}SRS \rangle. \qquad (*)$$

The cases $m = 1$ and $m = 2$ gave the trivial group but for $m = 3$, I obtained a group of order 120 which we shall show is $SL(2,5)$. The groups $(*)$ are also discussed in (19) where they are denoted by $Cam(m)$, following their introduction in (8).

From this class E. F. Robertson and I ((12), (16)) then decided to consider the class $\underset{\sim}{G}(m,n)$ where

$$\underset{\sim}{G}(m,n) \cong \langle a,b \mid [a^m,b^{-1}] = a^{-1}b^na, [b^m,a^{-1}] = b^{-1}a^nb \rangle.$$

The groups $(*)$ are thus the groups $\underset{\sim}{G}(m,1)$.

I then decided on one further generalisation, namely to consider the groups $\underset{\sim}{G}(l,m,n)$ where

$\underset{\sim}{G}(l,m,n) \cong \langle a,b \mid ab^m a^{-1} b^{-1} a^{-n} b^{l-l} = ba^m b^{-1} a^{-1} b^{-n} a^{l-l} = 1 \rangle$.

The groups $\underset{\sim}{G}(m,n)$ are just the groups $\underset{\sim}{G}(m,m,n)$.

It is interesting to note that this class of groups in fact includes the group that was the original source of interest together with the Benson and Mendelsohn generalisation, that is the groups (†) or, in the new notation, $\underset{\sim}{G}(n+1,n,0)$. The chapter includes in section 3.3 a proof that if $(l,m) \neq 1$, $\underset{\sim}{G}(l,m,0)$ is infinite and if $(l,m) = 1$ then $\underset{\sim}{G}(l,m,0)$ is metacyclic of order $|l-m|^3$.

The question arises as to why the groups $\underset{\sim}{G}(l,m,n)$ might be interesting? Where the groups $\underset{\sim}{G}(l,m,n)$ are finite, then we have cyclically presented finite 2-generator groups of deficiency zero. In the notation of section 1.3 they are groups of type $G_2(w)$ where $w = ab^m a^{-1} b^{-1} a^{-n} b^{l-l}$. As mentioned in chapter 1 the class of cyclically presented finite groups of deficiency zero known to have 2-generator 2-relation presentations is small.

In this chapter we consider the groups $\underset{\sim}{G}(l,m,n)$ giving first some isomorphisms between groups and some homomorphisms that occur. Then we give some results on those groups where we have been able to determine their order, either finite or infinite. Of note is the fact that the modified Todd-Coxeter algorithm is used to show that some of the groups $\underset{\sim}{G}(l,m,n)$ are infinite. Six groups, namely $\underset{\sim}{G}(2,2,3)$, $\underset{\sim}{G}(2,2,-3)$, $\underset{\sim}{G}(-1,-1,4)$, $\underset{\sim}{G}(2,3,-2)$, $\underset{\sim}{G}(-2,2,-1)$ and $\underset{\sim}{G}(-2,3,-1)$ are shown to be finite non-metacyclic groups of deficiency zero. These groups have orders $2^{15}.3^3$, $2^8.3^3$, $2^9.3.5$, $2^3.3^3.7$, $2^3.3.5.11$

and $2^3.3^6$ respectively. We also give an infinite series of non-metacyclic groups all having the alternating group A_5 as a homomorphic image.

Some computer results are given and the chapter concludes with tables showing, where known, the orders of the groups $\underset{\sim}{G}(l,m,n)$, $-2 \leq l \leq 5$, $-4 \leq m$, $n \leq 5$. In the tables we give the order of the groups, where known, and also include where appropriate the number of one of the theorems or lemmas from which the order may be obtained.

3.2 Isomorphisms and homomorphisms.

In this section we derive some isomorphisms between various subclasses of the groups $\underset{\sim}{G}(l,m,n)$. We also show that under certain conditions the groups $\underset{\sim}{G}(l,m,n)$ may have PSL(2,p), the 2-dimensional projective special linear group, as a homomorphic image. The proofs are all straightforward.

We show first that the groups $\underset{\sim}{G}(m,n)$ of (12), (16) are a special case of the groups $\underset{\sim}{G}(l,m,n)$.

<u>Lemma 3.2.1.</u> $\underset{\sim}{G}(m,n) \cong \underset{\sim}{G}(m,m,n)$.

<u>Proof.</u> $\underset{\sim}{G}(l,m,n)$ has a presentation

$$\langle a,b \mid ab^m a^{-1} b^{-1} a^{-n} b^{l-l} = ba^m b^{-1} a^{-1} b^{-n} a^{l-l} = 1 \rangle.$$

If we take $l = m$, then

$$\underset{\sim}{G}(m,m,n) \cong \langle a,b \mid [a^m,b^{-1}] = a^{-1}b^n a,\ [b^m,a^{-1}] = b^{-1}a^n b \rangle$$

$$\cong \underset{\sim}{G}(m,n).$$

<u>Lemma 3.2.2.</u> $\underset{\sim}{G}(0,m,n) \cong \underset{\sim}{G}(0,-m,-n)$,

$$\cong \underset{\sim}{G}(0,-n,-m),$$

$$\cong \underset{\sim}{G}(0,n,m).$$

<u>Proof.</u> $\underset{\sim}{G}(0,m,n)$ has a presentation

$$\langle a,b \mid ab^m a^{-1} b^{-1} a^{-n} b = ba^m b^{-1} a^{-1} b^{-n} a = 1 \rangle$$

which may be written as

$$\langle a,b \mid ab^{-m} a^{-1} b^{-1} a^n b = ba^{-m} b^{-1} a^{-1} b^n a = 1 \rangle$$

and this is a presentation for $\underset{\sim}{G}(0,-m,-n)$. Therefore $\underset{\sim}{G}(0,m,n) \cong \underset{\sim}{G}(0,-m,-n)$.

Now consider the map ϕ such that $a\phi = b^{-1}$, $b\phi = a^{-1}$. Then ϕ is an isomorphism and it is immediate that $\underset{\sim}{G}(0,m,n) \cong \underset{\sim}{G}(0,-n,-m)$. Combining the first two isomorphisms we obtain the third isomorphism $\underset{\sim}{G}(0,m,n) \cong \underset{\sim}{G}(0,n,m)$.

<u>Lemma 3.2.3.</u> $\underset{\sim}{G}(1,m,n) \cong \underset{\sim}{G}(1,1-n,1-m)$.

<u>Proof.</u> As in Lemma 3.2.2 consider the map ϕ such that $a\phi = b^{-1}$, $b\phi = a^{-1}$. Under the map ϕ the presentation for $\underset{\sim}{G}(1,m,n)$ becomes

$$\langle a,b \mid b^{-1} a^{-m} bab^n = a^{-1} b^{-m} aba^n = 1 \rangle,$$

that is

$$\langle a,b \mid ab^{1-n} a^{-1} b^{-1} a^{m-1} = ba^{1-n} b^{-1} a^{-1} b^{m-1} = 1 \rangle,$$

and since the map ϕ induces an isomorphism $\underset{\sim}{G}(1,m,n) \cong \underset{\sim}{G}(1,1-n,1-m)$.

<u>Lemma 3.2.4.</u>
$$\begin{aligned} \underset{\sim}{G}(l,1,n) &\cong \underset{\sim}{G}(l,n+l,1-l), \\ &\cong \underset{\sim}{G}(1-n,1,1-l), \\ &\cong \underset{\sim}{G}(1-n,2-l-n,n). \end{aligned}$$

<u>Proof.</u> $\underset{\sim}{G}(l,1,n) \cong \langle a,b \mid aba^{-1}b^{-1}a^{-n}b^{1-l} = bab^{-1}a^{-1}b^{-n}a^{1-l} = 1 \rangle$. From the relations $aba^{-1}b^{-1}a^{-n}b^{1-l} = 1$ and $bab^{-1}a^{-1}b^{-n}a^{1-l} = 1$

$$a^{-n}b^{1-l} = a^{l-1}b^n, \tag{1}$$

that is

$$a^{n+l-1}b^{n+l-1} = 1. \tag{2}$$

But the relations of $G(l,1,n)$ which may, by (1), be written in the form

$$aba^{-1}b^{-1}a^{l-1}b^{1-l}b^{n+l-1} = 1,$$

$$bab^{-1}a^{-1}b^{l-1}a^{1-l}a^{n+l-1} = 1,$$

become, using (2),

$$ab^{n+l-1}ba^{-1}b^{-1}a^{l-1}b^{1-l} = 1, \tag{3}$$

$$ba^{n+l}b^{-1}a^{-1}b^{l-1}a^{1-l} = 1. \tag{4}$$

Thus,

$$G(l,1,n) \text{ is a homomorphic image of } G(l,n+l,1-l). \tag{5}$$

A similar proof shows that it is an isomorphism

Alternatively, the relations for $\underset{\sim}{G}(l,1,n)$ may be written in the form

$$bab^{-1}a^{-1}b^{l-1}a^{n} = 1, \tag{6}$$

and

$$aba^{-1}b^{-1}a^{l-1}b^{n} = 1 \tag{7}$$

respectively. Thus

$$G(l,1,n) \text{ is a homomorphic image of } G(1-n,1,1-l). \tag{8}$$

A similar proof shows that it is an isomorphism.

Combining (5) and (8) we obtain the third isomorphism

$$\underset{\sim}{G}(l,1,n) \cong \underset{\sim}{G}(1-n,2-l-n,n).$$

<u>Corollary.</u> $\underset{\sim}{G}(l,1,1) \cong \underset{\sim}{G}(0,l-1,-1)$.

<u>Proof.</u> From Lemma 3.2.4 $\underset{\sim}{G}(l,1,1) \cong \underset{\sim}{G}(0,1,1-l)$. The result now follows from Lemma 3.2.2.

<u>Lemma 3.2.5.</u> $\underset{\sim}{G}(1-n,m,n) \cong \underset{\sim}{G}(2-n-m,2-m,m+n-1)$.

<u>Proof.</u> In Lemma 3.3.14 we shall show that in the usual presentation for $\underset{\sim}{G}(1-n,m,n)$ $a^{m-1}b^{m-1} = 1$, that is a^{m-1} and b^{m-1} are in the centre of $\underset{\sim}{G}(1-n,m,n)$. The two relations for $\underset{\sim}{G}(1-n,m,n)$ may be written as

$$ab^{2m-2}b^{2-m}a^{-1}b^{-1}a^{-n}b^{n} = 1, \tag{1}$$

$$ba^{2m-2}a^{2-m}b^{-1}a^{-1}b^{-n}a^{n} = 1, \tag{2}$$

and since a^{2m-2} and b^{2m-2} are in the centre of $G(1-n,m,n)$

$$ab^{2-m}a^{-1}b^{-1}b^{2m-2}a^{-n}b^{n} = 1, \tag{3}$$

$$ba^{2-m}b^{-1}a^{-1}a^{2m-2}b^{-n}a^{n} = 1. \tag{4}$$

Again using the fact that $a^{m-1}b^{m-1} = 1$, (3) and (4) may be written as

$$ab^{2-m}a^{-1}b^{-1}a^{1-m-n}b^{m+n-1} = 1, \tag{5}$$

$$ba^{2-m}b^{-1}a^{-1}b^{1-m-n}a^{m+n-1} = 1. \tag{6}$$

Thus $G(1-n,m,n)$ is a homomorphic image of $G(2-n-m,2-m,m+n-1)$. A similar argument yields the result.

Lemma 3.2.6. $G(1,m,n)/G'(1,m,n)$ is infinite if, and only if, $m-1 = \pm n$. Otherwise $G(1,m,n)/G'(1,m,n)$ has order $|(m-1-n)(m-1+n)|$. In particular $G(m,m,n)/G'(m,m,n)$ is isomorphic to $\mathbb{Z}_n \times \mathbb{Z}_n$ if $n \geq 1$ and to $\mathbb{Z}_{-n} \times \mathbb{Z}_{-n}$ if $n \leq 1$.

Proof. $G(1,m,n) \cong \langle a,b \mid ab^m a^{-1}b^{-1}a^{-n}b^{1-1} = ba^m b^{-1}a^{-1}b^{-n}a^{1-1} = 1\rangle$ so that $G(1,m,n)/G'(1,m,n)$ has a presentation

$$\langle a,b \mid a^{m-1} = b^n, b^{m-1} = a^n, [a,b]=1\rangle$$

from which the results are immediate.

In the next theorem we shall extend Theorem '1' of (16) and find conditions under which the group $G(1,m,n)$ has a simple homomorphic image. Implicit in the statement of the theorem is Euler's quadratic residue theorem, see for example Hardy and Wright (27), which states that if p is a prime, $p \equiv 1 \mod 4$, then there is an i such that $i^2 \equiv -1 \mod p$.

Theorem 3.2.7. Let p be a prime. Suppose $p \equiv 1 \mod 4$ and $i^2 \equiv -1 \mod p$. Then $G(1,m,n)$ has $PSL(2,p)$ as a homomorphic image whenever

$$1 \equiv 1 + i \mod p, \quad m \equiv 1 + i \mod p, \quad n \equiv -2i \mod p. \quad (*)$$

If l,m and n satisfy (*) and n is odd then $\underset{\sim}{G}(l,m,n)$ has $SL(2,p)$ as a homomorphic image.

Proof. Let $A = \begin{pmatrix} -1 & -1 \\ . & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & . \\ 1+i & -1 \end{pmatrix}$ where the entries in the matrices are in $GF(p)$. Then A and B generate $SL(2,p)$.

If n is odd and l,m,n satisfy (*) then

$$B^{1-l}AB^m = \begin{pmatrix} 1-2i & 1 \\ 1+i & i \end{pmatrix} = A^nBA.$$

Similarly $A^{1-l}BA^m = B^nAB$. Hence $\underset{\sim}{G}(l,m,n)$ has $SL(2,p)$ as a homomorphic image.

If n is even and l,m,n satisfy (*) then

$$B^{1-l}AB^m = \begin{pmatrix} 1-2i & 1 \\ 1+i & i \end{pmatrix} = -A^nBA.$$

Also $A^{1-l}BA^m = -B^nAB$. Thus, in this case, $\underset{\sim}{G}(l,m,n)$ has $PSL(2,p)$ as a homomorphic image.

In (12) it was shown that $\underset{\sim}{G}(3,3,1)$ is isomorphic to $SL(2,5)$. Since the order of $\underset{\sim}{G}(3,3,1)$ is 120, see for example (8) or (19), the result may also be obtained from Theorem 3.2.7. Finally let us consider the case when $l = m = -1$. It is easy to see that $\underset{\sim}{G}(-1,-1,n)$ may have $PSL(2,5)$ as a homomorphic image. For, using Theorem 3.2.7, it is seen that $i = 3$ gives $l = m = -1$ and $n \equiv 4 \bmod 5$. Moreover these are the only groups in the series $\underset{\sim}{G}(-1,-1,n)$ shown by Theorem 3.2.7 to have $PSL(2,p)$ as a homomorphic image, since $l = m = -1$ implies $i \equiv -2 \bmod p$. Hence $i^2 \equiv 4 \bmod p$. But $i^2 \equiv -1 \bmod p$ and thus $p = 5$.

3.3 Lemmas on orders of the groups $\underset{\sim}{G}(l,m,n)$.

In this section we show how the orders of some of the groups $\underset{\sim}{G}(l,m,n)$ may be determined. We consider first some finite groups and then some infinite groups. The section continues by considering examples where (l,m,n) groups, see section 1.4, are involved. We conclude by looking at the groups $\underset{\sim}{G}(1-n,m,n)$.

Lemma 3.3.1. $\underset{\sim}{G}(1,m,1) \cong \mathbb{Z}_{|m^2-2m|}$.

Proof. $\underset{\sim}{G}(1,m,1) \cong \langle a,b \mid b^{m-1} = a,\ a^{m-1} = b \rangle$,

that is

$$\underset{\sim}{G}(1,m,1) \cong \langle a \mid a^{m^2-2m} = 1 \rangle.$$

Lemma 3.3.2. $\underset{\sim}{G}(1-n,2,n) \cong \mathbb{Z}_{|2n+1|}$.

Proof. $\underset{\sim}{G}(1-n,2,n) \cong \langle a,b \mid ab^2a^{-1}b^{-1}a^{-n}b^n = 1,\ ba^2b^{-1}a^{-1}b^{-n}a^n = 1 \rangle$. The two relations give $ab = 1$ and so $\underset{\sim}{G}(1-n,2,n)$ is cyclic. It is then immediate that $\underset{\sim}{G}(1-n,2,n) \cong \mathbb{Z}_{|2n+1|}$.

Lemma 3.3.3. If $(l,n) = 1$, $\underset{\sim}{G}(l,0,n)$ has order $l^2 - n^2$. Otherwise $\underset{\sim}{G}(l,0,n)$ is infinite.

Proof. $\underset{\sim}{G}(l,0,n) \cong \langle a,b \mid b^n = a^l,\ a^n = b^l \rangle$.

Suppose $(l,n) = d \neq 1$, then $\underset{\sim}{G}(l,0,n)$ has the infinite group $\mathbb{Z}_d * \mathbb{Z}_d$, see 1.3, as a homomorphic image.

Now $b^{n^2} = a^{ln} = b^{l^2}$, that is $b^{l^2-n^2} = 1$. If $(l,n) = 1$ then a is a power of b and so the group is cyclic. In this case $\underset{\sim}{G}(l,0,n)$ is cyclic of order $l^2 - n^2$.

Lemma 3.3.4. $\underset{\sim}{G}(l,-1,n)$ is infinite if $(l-1,n-2) \geq 4$.

<u>Proof</u>. $\underset{\sim}{G}(l,-1,n) \cong \langle a,b \mid ab^{-1}a^{-1}b^{-1}a^{-n}b^{l-1} = ba^{-1}b^{-1}a^{-1}b^{-n}a^{l-1} = 1\rangle$ and so if we add the relation $(ab)^2 = 1$, then $\underset{\sim}{G}(l,-1,n)$ has H as a homomorphic image where

$$H \cong \langle a,b \mid a^{2-n}b^{l-1} = 1,\ b^{2-n}a^{l-1} = 1,\ (ab)^2 = 1\rangle.$$

Thus if $d = (l-1,n-2)$, $\underset{\sim}{G}(l,-1,n)$ has the group $(2,d,d)$, see 1.4, as a homomorphic image and this is infinite if $d \geq 4$.

<u>Theorem 3.3.5</u>. If $n \geq 1$, $\underset{\sim}{G}(1,1,n)$ is a finite metacyclic group of order n^3 and exponent n^2, the centre of $\underset{\sim}{G}(1,1,n)$ being cyclic of order n and equal to the derived group $\underset{\sim}{G}'(1,1,n)$. Further $\underset{\sim}{G}(1,1,-n)$ is isomorphic to $\underset{\sim}{G}(1,1,n)$.

<u>Proof</u>. For $n \geq 1$,

$$\underset{\sim}{G}(1,1,n) \cong \langle a,b \mid a^{1-n}b = ba,\ b^{1-n}a = ab\rangle.$$

Hence $b^{-1}a^{1-n}a^{-1}b^{1-n} = ab^{-1}ba^{-1} = 1$ and so $a^n = b^{-n}$. Thus $a^n \in Z(\underset{\sim}{G}(1,1,n))$, the centre of $\underset{\sim}{G}(1,1,n)$. Since $\underset{\sim}{G}(1,1,n)/\langle a^n\rangle \cong \mathbb{Z}_n \times \mathbb{Z}_n$ and since by Lemma 3.2.6 $\underset{\sim}{G}(1,1,n)/\underset{\sim}{G}'(1,1,n) \cong \mathbb{Z}_n \times \mathbb{Z}_n$ we must have $\langle a^n\rangle = \underset{\sim}{G}'(1,1,n)$. Now

$$bab^{-1} = b^n a = a^{1-n},$$

and so

$$b^n a b^{-n} = a^{1-n^2}.$$

Since $b^n a b^{-n} = a$, $a^{n^2} = 1$. Therefore $\underset{\sim}{G}(1,1,n)$ has order n^3 and is metacyclic of exponent n^2. If $n \leq -1$ a similar argument holds.

Alternatively we may show that $\underset{\sim}{G}(1,1,n)$ is a finite metacyclic group of order n^3 by using Theorem 3.3.9 where it is shown that if $(l,m) = 1$ the group with presentation

$$\langle a,b \mid ab^m = b^l a,\ ba^m = a^l b\rangle$$

is metacyclic of order $|l-m|^3$.

<u>Lemma 3.3.6</u>. $\underset{\sim}{G}(l,m,n)$ is infinite if $(l,m,n) \neq 1$.

<u>Proof</u>. If $(l,m,n) = d \neq 1$, then $\underset{\sim}{G}(l,m,n)$ has as a homomorphic image the group H where

$$H \cong \langle a,b \mid ab^m a^{-1} b^{-1} a^{-n} b^{1-l} = 1,\ ba^m b^{-1} a^{-1} b^{-n} a^{1-l} = 1, a^d = b^d = 1\rangle.$$

Thus $\underset{\sim}{G}(l,m,n)$ has $\mathbb{Z}_d * \mathbb{Z}_d$, see 1.3, as a homomorphic image and so is infinite.

In the next two results we consider the special case of the groups $\underset{\sim}{G}(m,m,n)$.

<u>Theorem 3.3.7</u>. $\underset{\sim}{G}(2,2,2n)/\langle a^2,b^2\rangle$ is isomorphic to D_∞, the infinite dihedral group.

<u>Proof</u>. $\underset{\sim}{G}(2,2,2n) \cong \langle a,b \mid ab^2a^{-1}b^{-1}a^{-2n}b^{-1} = 1,\ ba^2b^{-1}a^{-1}b^{-2n}a^{-1} = 1\rangle$. If we add the relations $a^2 = b^2 = 1$, then $\underset{\sim}{G}(2,2,2n)$ has as a homomorphic image

$$\langle a,b \mid a^2 = 1,\ b^2 = 1\rangle$$

that is $\mathbb{Z}_2 * \mathbb{Z}_2$ or D_∞, the infinite dihedral group. To show that $\underset{\sim}{G}(2,2,2n)/\langle a^2,b^2\rangle$ is isomorphic to D_∞ we require to show further that $\langle a^2,b^2\rangle$ is normal in $\underset{\sim}{G}(2,2,2n)$. It is sufficient to prove that $ab^2a^{-1} \in \langle a^2,b^2\rangle$ for then, by symmetry, $ba^2b^{-1} \in \langle a^2,b^2\rangle$. But

$$\begin{aligned} b^2 &= a^{-1}ba^{2n}ba, \\ &= a^{-1}(ba^2b^{-1})^n b^2 a, \\ &= a^{-1}(ab^{2n}a)^n b^2 a, \\ &= (b^{2n}a^2)^{n-1} b^{2n}(ab^2a^{-1})a^2. \end{aligned}$$

Hence

$$ab^2a^{-1} = b^{-2n}(b^{2n}a^2)^{1-n} b^2 a^{-2},$$

and so $ab^2a^{-1} \in \langle a^2,b^2\rangle$ as required.

<u>Theorem 3.3.8</u>. Let $(m,n) = 1$. Then $\underset{\sim}{G}(m,m,n)$ has $G(1,1,n)$ as a

homomorphic image. In particular $\underset{\sim}{G}(-n,-n,n+1)$ is isomorphic to $\underset{\sim}{G}(1,1,n+1)$.

Proof. Consider the group

$$H \cong \langle a,b \mid ab^m a^{-1} b^{-1} a^{-n} b^{1-m} = 1,\ ba^m b^{-1} a^{-1} b^{-n} a^{1-m} = 1,\ a^n b^n = 1 \rangle.$$

H is obviously a homomorphic image of $\underset{\sim}{G}(m,m,n)$. Since $a^n b^n = 1$, a^n and b^n are in $Z(H)$, the centre of H, and the relation $ab^m a^{-1} b^{-1} a^{-n} b^{1-m} = 1$ becomes $ab^m a^{-1} = b^m b^{-n}$, from which $a^n b^m a^{-n} = b^m b^{-n^2}$ and so $b^{n^2} = 1$. But

$$\langle a,b \mid a^n = 1,\ b^n = 1,\ ab^m a^{-1} = b^m,\ ba^m b^{-1} = a^m \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_n,$$

since m is coprime to n^2 and so the relation $ab^m a^{-1} = b^m$ implies that a and b commute. ~~Therefore~~ In fact H', the derived group of H, $= Z(H) = \langle a^n \rangle$ and H is isomorphic to $\underset{\sim}{G}(1,1,n)$ as may be checked by changing the generators.

Now consider

$$\underset{\sim}{G}(-n,-n,n+1) \cong \langle a,b \mid ab^{-n} a^{-1} b^{-1} a^{-n-1} b^{n+1} = 1,\ ba^{-n} b^{-1} a^{-1} b^{-n-1} a^{n+1} = 1 \rangle.$$

The first relation gives $b^{n+1} ab^{-n} a^{-1} = a^{n+1} b$ and the second gives $a^{n+1} ba^{-n} b^{-1} = b^{n+1} a$. Thus $b^{n+1} ab^{-n} a^{-1} = b^{n+1} aba^n$, that is $a^{-(n+1)} = b^{n+1}$. Therefore by the first part of the theorem $\underset{\sim}{G}(-n,-n,n+1)$ is isomorphic to $\underset{\sim}{G}(1,1,n+1)$.

The next result is a generalisation of a result in (3) in which $l = m+1$, this itself being a generalisation of a result in (7) where we have $l = 3$, $m = 2$. In my proof in (7) I made use of the modified Todd-Coxeter coset enumeration algorithm and this proof was modified to give the algebraic proof in (3). However the proof that now follows for the general case is direct, being that of (17).

Theorem 3.3.9. $\underset{\sim}{G}(l,m,0)$ is infinite if $(l,m) \neq 1$ and is metacyclic of order $|l-m|^3$ if $(l,m) = 1$.

<u>Proof.</u> $\underset{\sim}{G}(l,m,0) \cong \langle a,b \mid ab^m = b^l a,\ ba^m = a^l b\rangle$.

We can assume without loss of generality that $l \geq m$. There are two cases to consider.

(i) If $(l,m) = d \neq 1$ then adding the relations $a^d = b^d = 1$ to $\underset{\sim}{G}(l,m,0)$ shows that $\underset{\sim}{G}(l,m,0)$ has $\mathbb{Z}_d * \mathbb{Z}_d$, see 1.3, as a homomorphic image. Therefore $\underset{\sim}{G}(l,m,0)$ is infinite.

(ii) If $(l,m) = 1$, then the relation $ab^m a^{-1} = b^l$ gives, for any i,

$$a^i b^{m^i} a^{-i} = b^{l^i}. \qquad (1)$$

With $i = m$ in (1) and conjugating by b^{-1} we obtain

$$ba^m b^{m^m} a^{-m} b^{-1} = b^{l^m}$$

and so

$$a^l b^{m^m} a^{-l} = b^{l^m}. \qquad (2)$$

However with $i = l$ in (1) we obtain

$$a^l b^{m^l} a^{-l} = b^{l^l}, \qquad (3)$$

and since

$$1 = a^l b^{m^l} b^{-m^m(m^{l-m})} a^{-l},$$

$$= b^{l^l} b^{l^m(-m^{l-m})},$$

we also have

$$b^{l^m(l^{l-m} - m^{l-m})} = 1. \qquad (4)$$

Raising (2) to the power $l^{l-m} - m^{l-m}$ we obtain $b^{(l^{l-m} - m^{l-m})} = 1$, since $(l,m) = 1$.

Now $(m, l^{l-m} - m^{l-m}) = 1$ so there exist integers α, β such that $\alpha m + \beta(l^{l-m} - m^{l-m}) = 1$. Then since $aba^{-1} = b^{\alpha l}$ gives

$$ab^m a^{-1} = b^{\alpha m l} = b^{l - \beta l(l^{l-m} - m^{l-m})} = b^l,$$

$$\underset{\sim}{G}(l,m,0) \cong \langle a,b \mid aba^{-1} = b^{\alpha l}, bab^{-1} = a^{\alpha l}, a^{(l^{l-m}-m^{l-m})} = b^{(l^{l-m}-m^{l-m})} = 1 \rangle.$$

Now $ab = b^{\alpha l}a$ and $ba = a^{\alpha l}b$ give $ba = a^{\alpha l-1}b^{\alpha l}a$, that is $a^{\alpha l-1} = b^{1-\alpha l}$. Therefore from $ab = b^{\alpha l}a$ we obtain $b^{(\alpha l-1)^2} = 1$.

Hence the order of a and b is

$$\begin{aligned}
((\alpha l-1)^2,(l^{l-m}-m^{l-m})) &= (\alpha^2 l^2-2\alpha l+1,(l^{l-m}-m^{l-m})),\\
&= (m^2\alpha^2 l^2-2m^2\alpha l+m^2,(l^{l-m}-m^{l-m}))\\
&\quad \text{since } (m^2,(l^{l-m}-m^{l-m})) = 1,\\
&= ((l-m)^2,(l^{l-m}-m^{l-m})),\\
&= (l-m)^2,
\end{aligned}$$

since

$$\begin{aligned}
l^{l-m}-m^{l-m} &= (l-m)(l^{l-m-1}+l^{l-m-2}m+ \dots +lm^{l-m-2}+m^{l-m-1})\\
&= (l-m)\{(l^{l-m-1}-l^{l-m-1})+l^{l-m-2}(m-l)+l^{l-m-3}(m^2-l^2)+\dots\\
&\quad +l(m^{l-m-2}-l^{l-m-2})+m^{l-m-1}-l^{l-m-1}+(l-m)l^{l-m-1}\}\\
&= (l-m)^2\{-l^{l-m-2}-(l+m)^{l-m-3}-\dots-(l^{l-m-2}+ml^{l-m-3}+\dots m^{l-m-2})\\
&\quad +l^{l-m-1}\}.
\end{aligned}$$

Raising $a^{1-\alpha l} = b^{\alpha l-1}$ to the power m gives $a^{m-\alpha ml} = b^{\alpha ml-m}$ showing that $a^{m-l} = b^{l-m}$.

Therefore $\langle a \rangle$ is normal in $\underset{\sim}{G}(l,m,0)$, $|\underset{\sim}{G}(l,m,0)/\langle a \rangle| = l-m$ and $|\langle a \rangle| = (l-m)^2$. Therefore $\underset{\sim}{G}(l,m,0)$ is metacyclic of order $(l-m)^3$ as required.

The next four lemmas all involve (l,m,n) groups. These are discussed in section 1.4 but further discussion is also to be found in chapter 6 of (24).

Lemma 3.3.10. $\underset{\sim}{G}(\ell,m,n)$ is infinite if there exists an integer t such that $\ell \equiv 1 \bmod t$, $n \equiv 2 \bmod t$ and $(m+1,t) \geq 4$.

Proof. The relations for $\underset{\sim}{G}(1,m,n)$ may be written in the form

$$b^{m+1}b^{-1}a^{-1}b^{-1}a^{-n}b^{1-\ell}a = 1,$$

$$a^{m+1}a^{-1}b^{-1}a^{-1}b^{-n}a^{1-\ell}b = 1,$$

and if we add the relations $a^t = b^t = (ab)^2 = 1$, it is then immediate that $\underset{\sim}{G}(1,m,n)$ has as a homomorphic image H where

$$H \cong \langle a,b \mid a^{(t,m+1)} = b^{(m+1,t)} = (ab)^2 = 1 \rangle .$$

H is the (1,m,n) group (2,m+1,t) and this is infinite under the given conditions.

Lemma 3.3.11. $\underset{\sim}{G}(1,m,n)$ has $\langle -2,3|t \rangle$ as a factor group if $m \equiv 1 \bmod t$, $n \equiv 1 \bmod t$ and $1 \equiv 0 \bmod t$.

Proof. The result follows on adding the relations $a^t = b^t = 1$ and considering the properties of the $\langle -1,m|n \rangle$ groups as discussed in 1.4. $\underset{\sim}{G}(1,m,n)$ has a homomorphic image H where

$$H \cong \langle a,b \mid aba^{-1}b^{-1}a^{-1}b = 1, bab^{-1}a^{-1}b^{-1}a = 1, a^t = b^t = 1 \rangle$$

that is

$$H \cong \langle a,b \mid a^t = b^t = 1,\ aba = bab \rangle.$$

Thus, see 1.4, H is isomorphic to $\langle -2,3|t \rangle$.

Corollary. $\underset{\sim}{G}(1,m,n)$ is infinite if there exists $t \geq 6$ with $m \equiv 1 \bmod t$, $n \equiv 1 \bmod t$ and $1 \equiv 0 \bmod t$.

Proof. Since $\langle -2,3|t \rangle$ is infinite if $t \geq 6$ the result is immediate.

Lemma 3.3.12. $\underset{\sim}{G}(1,m,n)$ is infinite if there exists $t \geq 6$ with $m \equiv -1 \bmod t$, $n \equiv -1 \bmod t$ and $1 \equiv 0 \bmod t$.

Proof. The proof is similar to that of Lemma 3.3.11 and again we show that with the above conditions $\underset{\sim}{G}(1,m,n)$ has $\langle -2,3|t\rangle$ as a homomorphic image.

Lemma 3.3.13. $\underset{\sim}{G}(1,m,2)$ is infinite if either $m \geq 3$ or $m \leq -5$.

Proof. Add the relations $a^{m+1} = b^{m+1} = 1$, then $\underset{\sim}{G}(1,m,2)$ has as a homomorphic image H where

$$H \cong \langle a,b \mid aba = b^m,\ bab = a^m,\ a^{m+1} = b^{m+1} = 1\rangle ,$$

that is

$$H \cong \langle a,b \mid a^{m+1} = b^{m+1} = 1,\ (ab)^2 = 1\rangle$$

and this is the (l,m,n) group $(2,m+1,m+1)$ described in 1.4. Such a group is infinite if $m+1 \geq 4$ or $m+1 \leq -4$ from which the result follows.

The next lemma will prove useful in the theorems that complete section 3.3.

Lemma 3.3.14. In the groups $\underset{\sim}{G}(1-n,m,n)$ the relation $b^{m-1}a^{m-1} = 1$ holds.

Proof. $\underset{\sim}{G}(1-n,m,n) \cong \langle a,b \mid ab^ma^{-1}b^{-1}a^{-n}b^n = 1, ba^mb^{-1}a^{-1}b^{-n}a^n = 1\rangle$.

From the second relation $a^{-n}b^n = ba^mb^{-1}a^{-1}$, and therefore the first relation may be rewritten as

$$ab^ma^{-1}b^{-1}ba^mb^{-1}a^{-1} = 1,$$

that is

$$b^{m-1}a^{m-1} = 1,$$

as required. Therefore a^{m-1} and b^{m-1} are in the centre of $\underset{\sim}{G}(1-n,m,n)$.

Theorem 3.3.15. If $n \equiv 0 \bmod (m-1)$, $\underset{\sim}{G}(1-n,m,n)$ is metacyclic of order $|(m-1)^2(2n+m-1)|$.

Proof. We may assume that $m > 2$. The case $m = 2$ is considered in Lemma 3.3.2.

$$\underset{\sim}{G}(1-n,m,n) \cong \langle a,b \mid ab^m a^{-1} b^{-1} a^{-n} b^n = 1,\ ba^m b^{-1} a^{-1} b^{-n} a^n = 1 \rangle .$$

From Lemma 3.3.14 $b^{m-1}a^{m-1} = 1$. Since $n \equiv 0 \bmod (m-1)$ the relations for $\underset{\sim}{G}(1-n,m,n)$ may be rewritten as

$$aba^{-1}b^{-1}a^{1-m}a^{-n}a^{-n} = 1,$$
$$bab^{-1}a^{-1}b^{1-m}b^{-n}b^{-n} = 1,$$

that is

$$aba^{-1} = a^{2n+m-1}b,$$
$$bab^{-1} = b^{2n+m-1}a.$$

Thus $ab^{m-1}a^{-1} = a^{(m-1)(2n+m-1)}b^{m-1}$, from which $a^{(m-1)(2n+m-1)} = 1$. $\underset{\sim}{G}(1-n,m,n)/\langle a^{m-1}\rangle \cong \langle a,b \mid a^{m-1} = b^{m-1} = 1, ab = ba\rangle$ and so $\underset{\sim}{G}(1-n,m,n)/\langle a^{m-1}\rangle$ is a group of order $(m-1)^2$. Hence $\underset{\sim}{G}(1-n,m,n)$ is metacyclic of order $(m-1)^2(2n+m-1)$.

<u>Theorem 3.3.16.</u> (i) If $n \equiv -1 \bmod (m-1)$, $\underset{\sim}{G}(1-n,m,n)$ is met~~acyclic~~abelian of order $|(2n+m-1)(m-1)(2^{|m-1|}-1)|$.

(ii) If $n \equiv 1 \bmod (m-1)$, $\underset{\sim}{G}(1-n,m,n)$ has order

$$\begin{cases} 24|(2n+m-1)|(m-1)^2/(m+5)^2 & -4 \le m \le -1, \\ 24|(2n+m-1)|(m-1)^2/(m-7)^2 & 3 \le m \le 6, \end{cases}$$

and is infinite if $m \geq 7$ or $m \leq -5$.

<u>Proof.</u> (i) Since $n \equiv -1 \bmod (m-1)$ and since by Lemma 3.3.14 $a^{m-1}b^{m-1} = 1$, the presentation for $\underset{\sim}{G}(1-n,m,n)$

$$\langle a,b \mid ab^m a^{-1} b^{-1} a^{-n} b^n = 1,\ ba^m b^{-1} a^{-1} b^{-n} a^n = 1\rangle$$

may be rewritten as

$$\langle a,b \mid b^{-1}ab = a^{-1}bab^{-(2n+m+1)}, a^{-1}ba = b^{-1}aba^{-(2n+m+1)}, a^{m-1}b^{m-1} = 1\rangle.$$

Now

$$b^{-1}a^jb = a^{-1}b^ja\, b^{-j(2n+m+1)},$$

and so

$$a^{m-1} = b^{m-1}b^{-(m-1)(2n+m+1)},$$

from which

$$b^{(m-1)(2n+m-1)} = 1.$$

$$\underset{\sim}{G}(1-n,m,n)/\langle a^{m-1}\rangle \cong \langle a,b \mid a^{m-1}=1,\ b^{m-1}=1,\ a^{-1}ba = b^{-1}ab\rangle$$

Once we show that $\underset{\sim}{G}(1-n,m,n)/\langle a^{m-1}\rangle$ has order $|(m-1)(2^{m-1}-1)|$ it then follows that $\underset{\sim}{G}(1-n,m,n)$ is meta~~cyclic~~abelian of order $|(2n+m-1)(m-1)(2^{m-1}-1)|$. Let us therefore consider the group H where

$$H \cong \langle c,d \mid c^n = 1,\ d^n = 1,\ c^{-1}dc = d^{-1}cd\rangle.$$

Now $c(dc^{-1})c^{-1} = cdc^{-2} = dc^{-1}dc^{-1}$, and so $c(dc^{-1})c^{-1} = (dc^{-1})^2$. Similarly $c^2(dc^{-1})c^{-2} = c(dc^{-1})^2c^{-1} = (dc^{-1})^4$, and in general $c^n(dc^{-1})c^{-n} = (dc^{-1})^{2^n}$. Therefore $(dc^{-1})^{2^n-1} = 1$ and so H has order $n(2^n - 1)$.

(ii) Since $n \equiv 1 \bmod (m-1)$ and since by Lemma 3.3.14 $a^{m-1}b^{m-1} = 1$, the presentation for $\underset{\sim}{G}(1-n,m,n)$ may be rewritten as

$$\underset{\sim}{G}(1-n,m,n) \cong \langle a,b \mid aba^{-1}b^{-1}a^{-1}bb^{2n+m-3} = 1, bab^{-1}a^{-1}b^{-1}aa^{2n+m-3} = 1, a^{m-1}b^{m-1} = 1\rangle$$

and, as above,

$$b^{m-1} = a^{m-1}b^{-(m-1)(2n+m-3)}.$$

Thus

$$b^{(m-1)(2n+m-1)} = 1.$$

Now a^{m-1} and b^{m-1} are in $Z(\underset{\sim}{G}(1-n,m,n))$ and

$\underset{\sim}{G}(1-n,m,n)/\langle a^{m-1}\rangle \cong \langle a,b \mid a^{m-1}=1,\ b^{m-1}=1,\ aba = bab\rangle \cong H$ say.

Therefore $\underset{\sim}{G}(1-n,m,n)$ has H as a homomorphic image and has order $|(2n+m-1)||H|$.

Now H is the $\langle -1,m|n\rangle$ group $\langle -2,3|m-1\rangle$, see section 1.4, and this has order $\dfrac{24(m-1)^2}{(|m-1|-6)^2}$ if $2 \le |m-1| \le 5$, and is infinite when $|m-1| \ge 6$.

Thus we have the required order when $|m-1| \geq 2$. The case $m = 1$ does not occur, the case $m = 0$ is covered by Lemma 3.3.3 and the case $m = 2$ gives the trivial group.

Corollary. If $n \equiv 1 \bmod 5$, $\underset{\sim}{G}(1-n,6,n)$ has A_5 as a simple homomorphic image and so is not metabelian.

Proof. By the above theorem, when $n \equiv 1 \bmod 5$, $\underset{\sim}{G}(1-n,6,n)$ has $\langle -2,3|5\rangle$ as a homomorphic image. It is shown on page 74 of (24) that $\langle -2,3|5\rangle = (2,3,5) \times \mathbb{Z}_5$ and on page 68 of (24) that $(2,3,5)$ is isomorphic to A_5.

We therefore have an infinite series of cyclically presented non-metabelian, and so non-metacyclic, groups of deficiency zero.
In addition, if $n \equiv 1 \bmod 4$ $\underset{\sim}{G}(1-n,5,n)$ is also not metabelian.

3.4 Examples and computer results.

In this section we begin by showing how the modified Todd-Coxeter algorithm may be used to prove that some of the groups $\underset{\sim}{G}(1,m,n)$ are infinite. We then give a lemma that is useful in determining the orders of some of the specific groups discussed later in the section. The examples we then consider are the six finite non-metacyclic groups of deficiency zero mentioned in 3.4, namely the groups $\underset{\sim}{G}(2,2,-3)$, $\underset{\sim}{G}(2,2,3)$, $\underset{\sim}{G}(-1,-1,4)$, $\underset{\sim}{G}(2,3,-2)$, $\underset{\sim}{G}(-2,2,-1)$ and $\underset{\sim}{G}(-2,3,-1)$. Finally we give the other groups $\underset{\sim}{G}(1,m,n)$ whose orders we have been able to determine, in most cases by use of the coset enumeration programme (1).

In Lemma 3.3.7 we showed that the subgroup $\langle a^2,b^2\rangle$ is normal in $\underset{\sim}{G}(2,2,2n)$ and further that $\underset{\sim}{G}(2,2,2n)/\langle a^2,b^2\rangle$ is isomorphic to D_∞, the infinite dihedral group. By using the modified Todd-Coxeter algorithm to find a presentation for $\langle a,b^2\rangle$ we shall be able to obtain further information about the subgroup $\langle a^2,b^2\rangle$, the results being used in 3.4.9 when we discuss the groups $\underset{\sim}{G}(2,2,2)$ and $\underset{\sim}{G}(2,2,4)$. Now

$$\underset{\sim}{G}(2,2,2n) \cong \langle a,b \mid ab^{2n}aba^{-2}b^{-1} = 1,\ ba^{2n}bab^{-2}a^{-1} = 1\rangle.$$

Let $x = a, y = b^2$ and let $H = \langle x,y\rangle$. Assuming that $n \geq 1$ we obtain the following table, see sections 2.2 and 2.4,

$1a = x1$	$1b = 2$	$2b = y1$	$2a = 3$
$3a = xy^nx2$	$3b = 4$	$4b = yx^{2n}3$	$4a = 5$
$5a = xy^nxy^n4$	$5b = 6$	$6b = yx^{2n}(xy^nx)^n5$	$6a = 7$
	. . . .		

and in general, for $k > 1$,

$$(2k-1)a = w(2k-1).(2k-2), \qquad (2k-1)b = 2k,$$

$$2k\,b = w(2k).(2k-1), \qquad (2k)a = 2k+1,$$

where $w(k)$ is given by $w(1) = x$, $w(2) = y$, $w(3) = xy^nx$, $w(4) = yx^{2n}$ and, by induction, $w(k) = w(k-2)[w(k-3)]^n$; $k \geq 5$.

Each coset gives two relations for the subgroup $\langle x,y\rangle$, one of the relations being trivial. Denote by $R(k) = 1$ the non-trivial relation obtained from coset k. Then from the relation $ba^{2n}bab^{-2}a^{-1} = 1$ we get

$$R(1) = (xy^nx)^nyxy^{-1}x^{-1},$$

and from the relation $ab^{2n}aba^{-2}b^{-1} = 1$ we get

$$R(2) = (yx^{2n})^nxy^nxyx^{-2}y^{-1}.$$

By induction,

$$R(k) = [w(k+2)]^nw(k+1)w(k)[w(k-1)]^{-1}[w(k)]^{-1}; \quad k \geq 3.$$

Notice that if $n \leq 1$ the same argument holds. Thus we have the following theorem.

<u>Theorem 3.4.1.</u> If H is the subgroup $\langle a,b^2\rangle$ of $\underset{\sim}{G}(2,2,2n)$, then

$$H \cong \langle x,y \mid R(k) = 1,\ k = 1,2,3,\ \dots\rangle$$

where the $R(k)$ are given inductively as above.

(Note that the fact that this is a presentation for H follows from Theorem 2.3.4, that is Theorem 4 of (2)).

We also have the following lemma.

Lemma 3.4.2. If $H = \langle a, b^n \rangle$ is a subgroup of $G(-1,-1,n)$ then H is normal in $G(-1,-1,n)$ and $G(-1,-1,n)/H$ is isomorphic to $\mathbb{Z}_n$.

Proof. We obtain a presentation for H from the modified Todd-Coxeter algorithm. Define n cosets $1, 2, \ldots, n$ by $1 = \langle a, b^n \rangle$ and $ib = i+1$, $1 \le i \le n-1$. From the subgroup generators $y = b^n$ and $x = a$ we obtain $nb = y1$ and $1a = x1$. Define $w_1 = x$, $w_2 = x^{-1}y^{-1}x^n$, $w_i = w_{i-2}^n w_{i-1}$, $3 \le i \le n$. Then from the relation $aba^{-1}b^{-1}a^{-1}b^{-n}a = 1$ we obtain $2a = w_2 2$ and from the relation $bab^{-1}a^{-1}b^{-1}a^{-n}b = 1$ we obtain $ia = w_i i$, $3 \le i \le n$. The relations for H come from considering $i.aba^{-1}b^{-1}a^{-1}b^{-n}a = R(i).i$, $2 \le i \le n$ and $1.bab^{-1}a^{-1}b^{-1}a^{-n}b = S(1).1$, $n.bab^{-1}a^{-1}b^{-1}a^{-n}b = S(n).n$. Therefore

$$H \cong \langle x, y \mid R(i) = 1, 2 \le i \le n,\ S(1) = 1,\ S(n) = 1 \rangle,$$

that is

$$H \cong \langle x, y \mid y w_i w_{i+1} w_i^{-2} = 1; 2 \le i \le n-1, y w_n y w_1 y^{-1} w_n^{-2} = 1, w_n^n y w_1 w_2^{-1} y^{-1} = 1,$$

$$w_{n-1}^n w_n y w_1^{-1} y^{-1} = 1 \rangle.$$

3.4.3. The group $G(2,2,-3)$.

$$G(2,2,-3) \cong \langle a, b \mid ab^2a^{-1}b^{-1}a^3b^{-1} = 1,\ ba^2b^{-1}a^{-1}b^3a^{-1} = 1 \rangle.$$

Let H be the subgroup of $G(2,2,-3)$ generated by $[a^{-1}, b^{-1}]$, $[a^{-1}, b]$, $[a, b]$. Clearly $H \le G'(2,2,-3)$ and in fact the coset enumeration programme (1) shows that $|G(2,2,-3):H| = 18$. Since by Lemma 3.2.6

$|\underset{\sim}{G}(2,2,-3) : \underset{\sim}{G}'(2,2,-3)| = 9$, H is a subgroup of index 2 in $\underset{\sim}{G}'(2,2,-3)$. Let $x = [a^{-1},b^{-1}]$, $y = [a^{-1},b]$ and $z = [a,b]$. Then we may use the modified Todd-Coxeter algorithm to find a presentation for H. The relations could be obtained by hand but we used the Wilde programme (47), see 2.5, to find the words in the subgroup generators which give the relations between the coset representatives. We then used the programme CCRG, see 2.5, which obtained from these words a presentation for the subgroup H. The following twenty relations were obtained for H:

$$x^{-2}yz^{-1}y^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}xyzy^{-1}x^{-1}yz^{-1}x^{-1}yz = 1,$$

$$z^{-1}y^{-1}xzy^{-1}xy^{-1}x^{2}yx^{-1}yzy^{-1}x^{-1}yz^{-1}y^{-1}x^{2} = 1,$$

$$x^{-1}yzy^{-1}xy^{-1}x^{-1}yz^{-1}x^{-1}yzy^{-1}x^{-1}yz^{-1}x^{-1}yzx^{-2}yzy^{-1} = 1,$$

$$z^{-1}y^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}xyz^{-1}y^{-1}xy^{-1}x^{-1}yz^{-1}y^{-1}x^{2}z^{-1}y^{-1}x^{-1}yz^{-1}y^{-1}x^{2}z^{-1}y^{-1}$$
$$xzy^{-1}xyz^{-2}x^{-1}yzy^{-1}x^{-1}yz^{-1}x^{-1}yzx^{-2}yx^{-1}yzy^{-1}x^{-1}yz^{-1}x^{-1}yzx^{-2}yzy^{-1}xyz^{-1}$$
$$y^{-1}x^{2}z^{-1}y^{-1}xy^{-1}xy^{-1}x^{-1}yz^{-1}y^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}x^{-1}yz^{-1}x^{-1}yzx^{-2}yzy^{-1}x^{2}z^{-1}y^{-1}$$
$$xzy^{-1}x = 1,$$

$$x^{-1}yz^{-1}y^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}xyz^{-1}y^{-1}xz^{2}y^{-1}x^{-1}yz^{-1}x^{-1}yzx^{-2}yzy^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}$$
$$xz^{-1}y^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}xyz^{-1}y^{-1}xz^{2}y^{-1}x^{-1}yz^{-1}x^{-1}yzx^{-2}yzy^{-1}xyzx^{-2}yzy^{-1} = 1,$$

$$x^{-2}yz^{-1}y^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}xyz^{-1}y^{-1}xy^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}xyz^{-1}y^{-1}xz^{2}y^{-1}x^{-1}yz^{-1}$$
$$x^{-1}yzx^{-2}yzy^{-1}xyzx^{-1}yz^{-1}y^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}xy = 1,$$

$$yzx^{-1}yx^{-1}yz^{-1}x^{-1}yzx^{-2}yx^{-1}yz^{-1}x^{-1}yzx^{-2}yz^{-1}y^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}xyz^{-1}x^{-1}yzx^{-2}$$
$$yzy^{-1}x^{-1}yx^{-1}yz^{-1}x^{-1}yzy^{-1}x^{-1}yz^{-1}x^{-1}yzx^{-2}yzx^{-1}yz^{-1}x^{-1}yzx^{-1} = 1,$$

$$y^{-1}xy^{-1}x^{-1}yz^{-1}x^{-1}yzx^{-2}yx^{-1}yzy^{-1}x^{-1}yz^{-1}x^{-1}yzx^{-2}yzy^{-1}xyz^{-1}y^{-1}x^{2}z^{-1}y^{-1}x = 1,$$

$$y^{-1}x^{-1}yz^{-1}x^{-1}yzx^{-2}yzx^{-1}yz^{-1}x^{-1}yzx^{-2}yx^{-1}yz^{-1}x^{-1}yzx^{-2}yz^{-1}y^{-1}x^{2}z^{-1}y^{-1}x$$
$$zy^{-1}xyz^{-1}x^{-1}yzx^{-2}yzy^{-1}x^{-1}yz^{-1}y^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}xyz^{-1}y^{-1}xzy^{-1}xyz^{-1}y^{-1}x^{2}$$
$$z^{-1}y^{-1}xy^{-1}xy^{-1}x^{-2}yx^{-1}yz^{-1}x^{-1}yz = 1,$$

$$z^{-1}yx^{-2}yz^{-1}y^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}xy = 1,$$

$$x^{-1}yzx^{-3}yz^{-1}y^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}xy = 1,$$

$$x^{-2}yx^{-1}yzy^{-1}x^{-1}yz^{-1}x^{-1}yzx^{-2}yzy^{-1}xz^{-1}y^{-1}x^{-1}yz^{-1}y^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}$$
$$xyz^{-1}x^{-1}yz = 1,$$

$$yx^{-1}yz^{-1}x^{-1}yzx^{-2}yz^{-1}y^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}xyz^{-2}x^{-1}yzy^{-1}x^{-1}yz^{-1}x^{-1}yzx^{-2}yzx^{-1}$$
$$yz^{-1}x^{-1}yzx^{-2}yz^{-1}y^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}xyz^{-1}y^{-1}xy^{-1}x^{-1}yz^{-1}y^{-1}x = 1,$$

$$x^{-1}yx^{-1}yz^{-1}x^{-1}yzx^{-2}yx^{-1}yz^{-1}x^{-1}yzx^{-2}yz^{-1}y^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}xyz^{-1}y^{-1}$$
$$xy^{-1}x^{-2}y = 1,$$

$$y^{-1}xy^{-1}x^{-1}yz^{-1}y^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}x^{-1}yz^{-1}x^{-1}yzx^{-2}yzy^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}xy^{-1}$$
$$x^{2}z^{-1}y^{-1}xzy^{-1}xz^{-1}y^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}xyz^{-1}y^{-1}xzy^{-1}xz^{-1}y^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}$$
$$xyz^{-1}y^{-1}xz = 1,$$

$$xyz^{-1}y^{-1}x^{2}zy^{-1}xy^{-1} = 1,$$

$$x^{-2}yz^{-1}y^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}xyz^{-1}y^{-1}xz^{2}y^{-1}x^{-1}yz^{-1}x^{-1}yzx^{-2}yzy^{-1}xyzx^{-2}yzy^{-1}$$
$$xyx^{-1}yx^{-1}yzx^{-2}yzy^{-1}x^{-1}yz^{-1}y^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}xyz^{-1}y^{-1}xy^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}$$
$$xyz^{-1}y^{-1}xz^{2}y^{-1}x^{-1}yz^{-1}x^{-1}yzx^{-2}yzy^{-1}xyzx^{-1}yx^{-1}yz^{-1}x^{-1}yzx^{-2}yx^{-1}yz^{-1}x^{-1}$$
$$yzx^{-2}yz^{-1}y^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}xyz^{-1}x^{-1}yzx^{-2}yzy^{-1}x^{-1} = 1,$$

$$x^{3}z^{-1}y^{-1}xzy^{-1}xyz^{-1}y^{-1}xz^{2}y^{-1}x^{-1}yz^{-1}y^{-1}xyx^{-1}yz^{-1}x^{-1}yzx^{-2}yx^{-1}yz^{-1}x^{-1}yz$$
$$x^{-2}yz^{-1}y^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}xyz^{-1}x^{-1}yzx^{-2}yzy^{-1}x^{-1}yz^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}xyz^{-1}$$
$$y^{-1} = 1,$$

$$xyz^{-1}x^{-1}yzx^{-2}yz^{-1}y^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}xyz^{-1}x^{-1}yzx^{-2}yzy^{-1}x^{-1}yz^{-1}y^{-1}x^{2}z^{-1}$$
$$y^{-1}xzy^{-1}xyz^{-1}y^{-1}xy^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}xyz^{-1}y^{-1}xz^{2}y^{-1}x^{-1}yz^{-1}x^{-1}yzx^{-2}yzy^{-1} = 1,$$

$$z^{-1}y^{-1}xzy^{-1}xy^{-1}xyz^{-1}y^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}x^{-1}yz^{-1}x^{-1}yzx^{-2}yzx^{-1}yz^{-1}x^{-1}yz$$
$$x^{-2}yz^{-1}y^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}xyz^{-1}x^{-1}yzx^{-2}yzy^{-1}x^{-1}yz^{-1}y^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}xyz^{-1}$$
$$y^{-1}xy^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}xyz^{-1}y^{-1}xz^{2}y^{-1}x^{-1}yz^{-1}x^{-1}yzx^{-2}yzy^{-1}xyzx^{-1}yz^{-1}y^{-1}$$
$$x^{2}z^{-1}y^{-1}xzy^{-1}xy = 1.$$

These relations are, not surprisingly, too lengthy to be dealt with by the programme (1). We therefore used the shorter relations to simplify the longer relations until the programme (1) could handle the coset enumeration. During the simplification six of the relations were found to be redundant. Thus we obtain a presentation for H on the generators x,y,z with the following relations:

$$x^{-2}yz^{-1}y^{-1}x^{2}y^{-1}zy = 1,$$
$$x^{-1}yzx^{-1}y^{-1}z = 1,$$
$$yz^{-1}y^{-1}x^{2}zy^{-1}xy^{-1}x = 1,$$

$$x^{-3}yzy^{-1}xy^{-1}x^{-1}yz^{-1}x^{-1}y^{2} = 1,$$
$$x^{-2}yz^{-1}y^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}xyz^{-1}y = 1,$$
$$xy^{-1}zy^{-2}x^{2}yx^{-1}yzy^{-1} = 1,$$
$$y^{2}xzy^{-1}x^{2}y^{-2}xzyx^{-1}yzxy^{-1}z = 1,$$
$$yx^{-1}yz^{-1}x^{-1}yzy^{-1}z^{-1}x^{-1}y^{2}x^{-2}yx^{-1}yz^{-1}x^{-1}yzy^{-2}xy^{-1}x^{-1}yz^{-1}y^{-1}x = 1,$$
$$y^{-1}z^{2}y^{-1}x^{-1}yz^{-1}x^{-1}yz = 1,$$
$$x^{-3}yz^{-1}x^{-1}y^{2}xy^{-1}zy^{-1}x^{-1}y = 1,$$
$$x^{-1}yx^{-1}yzy^{-1}x^{-1}yz^{-2}yx^{-1}yzy^{-2}zy^{-1}z^{-1}y = 1,$$
$$x^{-1}yx^{-1}yz^{-2}yx^{-1}y^{2}z^{-2}y^{-1}xy^{-1}x^{-2}y = 1,$$
$$xy^{-2}xzyz^{-1}y^{-1}xzy^{-1}xy^{-1}x^{2}y^{-2}xzyz^{-1}yx^{-1}yzy^{-1} = 1,$$
$$y^{-1}xy^{-1}xy^{-1}zy^{-1}x^{-1}yz^{-1}yz^{-1}y^{-1}xzy^{-1}xy^{-1}x^{2}z^{-1}y^{-1}xzy^{-1}xy^{-1}x^{2}y^{-2}xzy^{-1}$$
$$xy^{-1}x^{2}y^{-2}xz = 1.$$

The index of $\langle x \rangle$ in H is 32 and $\langle x \rangle$ is not a normal subgroup of H. Hence H is not abelian and since $H \leq \underset{\sim}{G}'(2,2,-3)$ the group $\underset{\sim}{G}(2,2,-3)$ is not metabelian. Coset enumeration gives $|H| = 2^{7}.3$ and so $|\underset{\sim}{G}(2,2,-3)| = 2^{8}.3^{3}$. By Theorem 3.3.8 $\underset{\sim}{G}(2,2,-3)$ has a homomorphic image isomorphic to $\underset{\sim}{G}(1,1,3)$, a group of order 27, and so $\underset{\sim}{G}(2,2,-3)$ is an extension of a 2-group by $\underset{\sim}{G}(1,1,3)$. Since $|\underset{\sim}{G}(2,2,-3):\langle a\rangle| = 2^{6}.3$ the order of a is 36. Hence a^{4} has order 9 and so a^{4} is contained in a Sylow 3-subgroup P. If $\underset{\sim}{G}(2,2,-3)$ is nilpotent P is normal in $\underset{\sim}{G}(2,2,-3)$. Let N be the normal closure of $\langle a^{4}\rangle$ in $\underset{\sim}{G}(2,2,-3)$. Since P is normal $N \leq P$ so $\underset{\sim}{G}(2,2,-3)/N$ has order divisible by 2^{8}. Add to the relations for $\underset{\sim}{G}(2,2,-3)$ the relation $a^{4} = 1$. Then coset enumeration gives $|\underset{\sim}{G}(2,2,-3):N| = 24$ and so $\underset{\sim}{G}(2,2,-3)$ is not nilpotent. Since $|\underset{\sim}{G}(2,2,-3)|$ is of the form $p^{\alpha}q^{\beta}$, where p,q are primes, $\underset{\sim}{G}(2,2,-3)$ is of course soluble.

3.4.4. The group $\underset{\sim}{G}(2,2,3)$.

In the ~~generalisation~~ modification of the groups $\langle R,S \mid RS^n = S^{n+1}R, SR^n = R^{n+1}S\rangle$ to the groups $\underset{\sim}{G}(m,n)$, see section 3.1, it soon became apparent from the coset enumeration programme (1) that $\underset{\sim}{G}(2,3)$ might well turn out to be of finite order, although of an order too great for the programme (1) to handle directly. We now show that $\underset{\sim}{G}(2,2,3)$ has in fact order $2^{15}.3^3$ and is neither metacyclic nor nilpotent.

$\underset{\sim}{G}(2,3)$ is isomorphic to $\underset{\sim}{G}(2,2,3)$ and so

$\underset{\sim}{G}(2,2,3) \cong \langle a,b \mid ab^2a^{-1}b^{-1}a^{-3}b^{-1} = 1, ba^2b^{-1}a^{-1}b^{-3}a^{-1} = 1\rangle.$

Since $aba = a^{-2}b^{-1}ab^2$ and $bab = b^{-2}a^{-1}ba^2$, $(ab)^3 = (a^{-2}b^{-1}ab^2)(b^{-2}a^{-1}ba^2) = 1$. The fact that $(ab)^3 = (ba)^3 = 1$ is of some help in the coset enumeration, see 2.6, each relation being of length 6. We know by Lemma 3.2.6 that the index of $\underset{\sim}{G}'(2,2,3)$ in $\underset{\sim}{G}(2,2,3)$ is 9. In fact $\underset{\sim}{G}(2,2,3)/\underset{\sim}{G}'(2,2,3) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

If H_1 is the subgroup $\langle [a,b],[a^{-1},b^{-1}],[a^{-1},b^2]\rangle$ then $|\underset{\sim}{G}(2,2,3):H_1| = 18$, if H_2 is the subgroup $\langle [a,b],[a^{-1},b],[a^{-1},b^2]\rangle$ then $|\underset{\sim}{G}(2,2,3):H_2| = 18$ and if H_3 is the subgroup $\langle [a,b],[a^{-1},b^{-1}],[a^{-1},b]\rangle$ then again $|\underset{\sim}{G}(2,2,3):H_3| = 18$. Similarly we use the coset enumeration programme (1) to find the indices of $K_1 = \langle [a,b],[a^{-1},b]\rangle$, $K_2 = \langle [a^{-1},b^{-1}],[a^{-1},b]\rangle$ and $K_3 = \langle [a,b],[a^{-1},b^{-1}]\rangle$ in $\underset{\sim}{G}(2,2,3)$. When a presentation for H_3 is obtained we can also find the indices of $L_1 = \langle [a,b]\rangle$, $L_2 = \langle [a^{-1},b]\rangle$ and $L_3 = \langle [a^{-1},b^{-1}]\rangle$ in H_3. This information we display in figure 1 below.

So far the information obtained gives us some feel for the group $\underset{\sim}{G}(2,2,3)$. For the actual determination of its order we proceed as follows. Let H be the subgroup H_3, that is $H = \langle [a,b], [a^{-1},b^{-1}], [a^{-1},b] \rangle$. Then H is a subgroup of index 18 in $\underset{\sim}{G}(2,2,3)$ and has index 2 in $\underset{\sim}{G}'(2,2,3)$. (With the further commutator $[a,b^2]$ we get the derived group.) With $x = [a,b]$, $y = [a^{-1},b^{-1}]$, $z = [a^{-1},b]$ the modified Todd-Coxeter algorithm gives us a presentation for H on x,y,z with sixteen relations which with redundancies and algebraic manipulation using one relation to simplify another lead to the eight relations

$$[x^2,y^2] = 1, \tag{1}$$

$$[y^2,z^2] = 1, \tag{2}$$

$$zyx^{-1}z^{-1}yx^{-1} = 1, \tag{3}$$

$$(yx)^2z^{-2} = 1, \tag{4}$$

$$y^{-3}zx^{-1}z^{-1}x^2y^2z^{-1}x^2y^{-1}zx^{-1} = 1, \tag{5}$$

$$y^2z^{-1}x^2y^{-1}zx^2y^2z^{-1}x^2yzx^2 = 1, \tag{6}$$

$$z^{-1}y^2zx^{-3}y^{-2}z^{-1}y^{-2}zx^{-1}y^{-2} = 1, \tag{7}$$

$$x^{-1}z^{-1}y^3z^{-1}y^2xz^{-1}yx^{-2}z = 1. \tag{8}$$

The subgroup $\langle x^2,y^2,z^2 \rangle$ is abelian. From (1) and (2) $[x^2,y^2] = 1$ and $[y^2,z^2] = 1$. It remains to show that $[x^2,z^2] = 1$. From (3)

$$zyx^{-1} = (yx^{-1})^{-1}z,$$

that is

$$zyx^{-1}zyx^{-1} = zyx^{-1}(yx^{-1})^{-1}z,$$

that is

$$(yx^{-1})^{-1}zzyx^{-1} = z^2,$$

which is

$$[z^2,yx^{-1}] = 1. \tag{9}$$

From (4)

$$z^2 = (yx)^2 \quad \text{and so} \quad [z^2, yx] = 1. \qquad (10)$$

From (9) and (10)

$$[z^2, x^2] = 1.$$

The relations of H are sufficiently complicated to make it very difficult to find a presentation of a subgroup L of H of index greater than two by the modified Todd-Coxeter algorithm. However if the subgroup of H is known to be abelian the simplification is substantial and so it might be reasonable to try to find a presentation for $\langle x^2, y^2, z^2 \rangle$. The index $|H : \langle x^2, y^2, z^2 \rangle|$ is 256 which the programme (47) cannot handle.

A possible way is to find a larger abelian subgroup of H. Our first attempt was to take the subgroup $\langle x^2, y^2, z^2, zx^2y^2z \rangle$ which we showed was abelian and of index 128. The programme (47) was unable to complete the problem even after 7 hours 15 minutes and 18 seconds, a lot of computing time even with a relatively slow programme. We therefore look for a bigger abelian subgroup.

Let $A = \langle x^2, y^2, z^2, yx^2y, yz^2y \rangle$. Then we show by using relations (1), (2), (3) and (4) that A is abelian. We know that x^2, y^2 and z^2 commute. From (4) $(yx)^2 = z^2$ and so $[x^2, (yx)^2] = 1$. Thus $[x^2, yxy^{-1}] = 1$ and so $[x^2, yx^2y^{-1}] = 1$. Therefore x^2 and yx^2y commute. Next we show that $[z^2, yx^2y] = 1$. We already know from (3) that $[z^2, yx^{-1}] = 1$. But $[z^2, yx^{-1}] = 1$ gives both $[z^2, yx^{-1}] = 1$ and $[z^2, xy^{-1}] = 1$. Also from (4) $[z^2, yx] = 1$ so $[z^2, yx^2y^{-1}] = 1$ and thus $[z^2, yx^2y] = 1$ as required. Now $[z^2, yx^2y] = 1$ implies $[x^2, yz^2y] = 1$. Since it is immediate that $[yx^2y, yz^2y] = 1$ it only remains to show that $[z^2, yz^2y] = 1$ to prove that A is abelian. But $yz^2y = y^2xyxy$ by (4) and

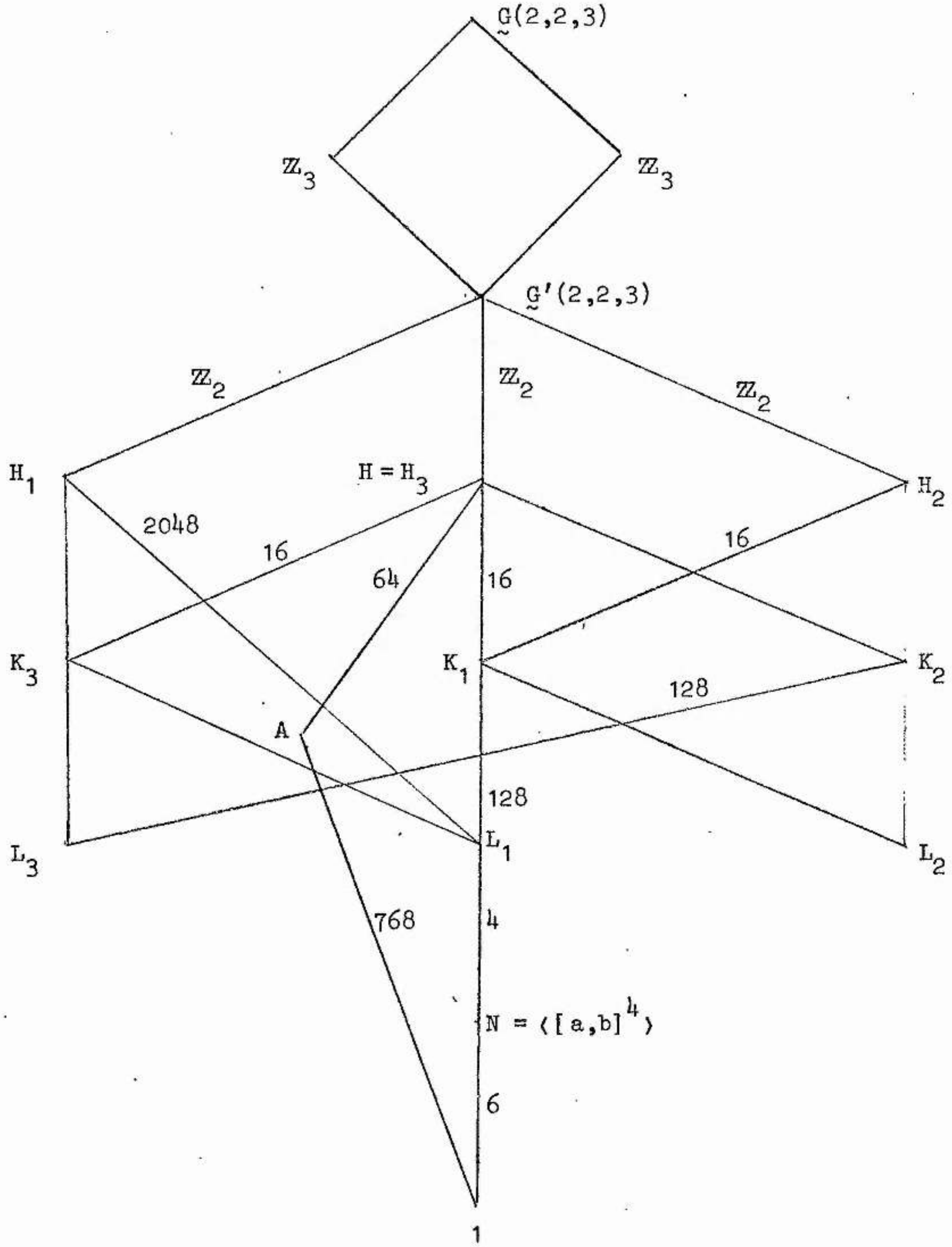
z^2 commutes with xy since we have shown that $[z^2, xy^{-1}] = 1$. Hence A is abelian.

Let $K = \langle x^2, y, z \rangle$. Then $H \geq K \geq A$ and $|H:K| = 2$. With $r = x^2$, $s = y$, $t = z$ we can find a presentation for K on r,s,t with fourteen relations. Since we know that $A = \langle r, s^2, t^2, srs, st^2s \rangle$ is abelian, we can simplify these relations to the following:

$$[r,s^2] = 1,$$
$$[r,t^2] = 1,$$
$$[r,srs] = 1,$$
$$[r,st^2s] = 1,$$
$$[s^2,t^2] = 1,$$
$$[r,ts^2t] = 1,$$
$$t^{-2}(srs^{-1}t^{-1}r)^2 = 1,$$
$$strs^2t^{-1}rs^{-1}trs^2t^{-1}r = 1,$$
$$s^{-3}t^{-1}stst^{-1}st = 1,$$
$$tsr^{-1}t^{-2}s^{-1}r^{-1}t^{-1}srt^{-2}s^{-1}r^{-1} = 1,$$
$$t^{-1}srsts^{-1}tsrst^{-1}rsr = 1,$$
$$t^{-1}s^{-2}t^{-1}r^{-1}st^{-1}st^{-2}sts^{-1}r^{-1} = 1,$$
$$s^{-1}ts^{-1}t^2r^{-1}s^{-2}t^{-1}s^{-1}tsr^{-1}t = 1,$$
$$s^3t^{-1}rst^{-1}s^{-1}t^{-1}rst^{-1} = 1.$$

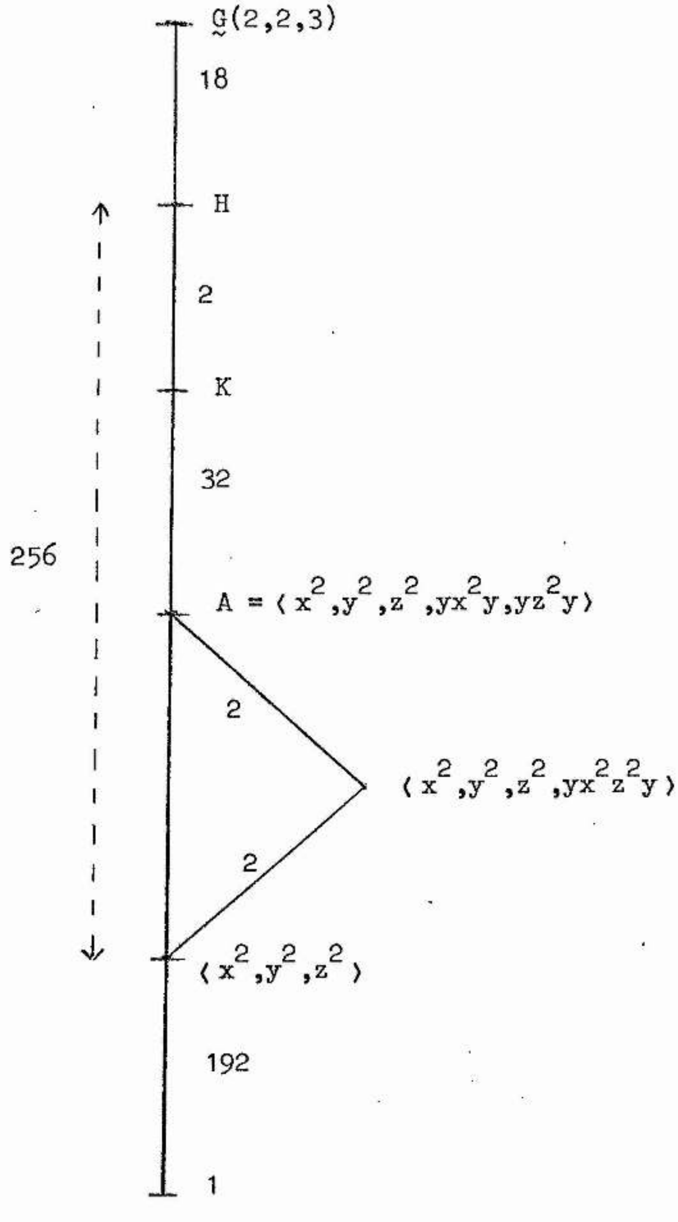
The index of A in K is 32. The modified Todd-Coxeter algorithm then gives a presentation for A. We first used the Wilde programme (47), see 2.5, the words obtained being too lengthy to reproduce here. However we know that A is abelian and so we abelianised these words by hand and then used them as input for CCRGAB, again see 2.5. With the relations for A thus obtained, and knowing that A is abelian, we found that $r^{12} = (s^2)^{12} = (t^2)^{12} = (srs)^{12} = (st^2s)^4 = 1$ and used the programme (1) to show that $|A| = 768$. This shows that the order of $\underset{\sim}{G}(2,2,3)$ is $2^{15}.3^3$.

Note that H is not abelian since $\langle x \rangle$ is not normal in H and so $\underset{\sim}{G}(2,2,3)$ is not metabelian. Also $\underset{\sim}{G}(2,2,3)$ is not nilpotent for we know that ab is an element of order 3 and if we add on the relation $ab = 1$ to the relations for $\underset{\sim}{G}(2,2,3)$ we get the group $\langle a,b \mid a^3 = 1,\ b^3 = 1,\ ab = \overset{1}{\cancel{ba}} \rangle$, that is a group of order 3. Therefore the Sylow 3 - subgroup containing ab is not normal and so $\underset{\sim}{G}(2,2,3)$ is not nilpotent. The information obtained about $\underset{\sim}{G}(2,2,3)$ is displayed in figure 2.



$\underset{\sim}{G}(2,2,3)$
$\mathbb{Z}_3$
$\mathbb{Z}_3$
$\underset{\sim}{G}'(2,2,3)$
$\mathbb{Z}_2$
$\mathbb{Z}_2$
$\mathbb{Z}_2$
H_1
$H = H_3$
H_2
2048
16
64
16
16
K_3
K_1
K_2
128
A
128
L_3
L_1
L_2
768
4
$N = \langle [a,b]^4 \rangle$
6
1

Figure 1


$\underset{\sim}{G}(2,2,3)$
18
H
2
K
32
256
$A = \langle x^2, y^2, z^2, yx^2y, yz^2y \rangle$
2
$\langle x^2, y^2, z^2, yx^2z^2y \rangle$
2
$\langle x^2, y^2, z^2 \rangle$
192
1

<u>Figure 2</u>

3.4.5. The group $\underset{\sim}{G}(-1,-1,4)$.

$$\underset{\sim}{G}(-1,-1,4) \cong \langle a,b \mid ab^{-1}a^{-1}b^{-1}a^{-4}b^{2} = 1,\ ba^{-1}b^{-1}a^{-1}b^{-4}a^{2} = 1 \rangle.$$

Let H be the subgroup of $\underset{\sim}{G}(-1,-1,4)$ generated by a and b^2. Coset enumeration gives that H is a normal subgroup of index 2. Let $x = a$ and $y = b^2$. Then

$$H \cong \langle x,y \mid y^2x^{-3}yxy^{-1}x^{-2} = 1,\ y^2x^{-1}y^{-2}x^4y^2x = 1,$$
$$x^{-1}y^{-2}xy^{-2}xy^{-2}x^{-2}y^3xy^{-1} = 1 \rangle.$$

The coset enumeration programme (1) now shows that $|H| = 3840 = 2^8.3.5$ and so $|\underset{\sim}{G}(-1,-1,4)| = 7680$. But by Theorem 3.2.7 $\underset{\sim}{G}(-1,-1,4)$ has PSL(2,5) as a homomorphic image and it is therefore an extension of a group of order 2^7 by PSL(2,5). Since $\underset{\sim}{G}(-1,-1,4)$ has a simple homomorphic image it is of course not soluble and is therefore neither metacyclic nor nilpotent.

3.4.6. The group $\underset{\sim}{G}(2,3,-2)$.

$$\underset{\sim}{G}(2,3,-2) \cong \langle a,b \mid ab^3a^{-1}b^{-1}a^2b^{-1} = 1,\ ba^3b^{-1}a^{-1}b^2a^{-1} = 1 \rangle.$$

We shall show that $\underset{\sim}{G}(2,3,-2)$ is a group of order 1512. The coset enumeration programme (1) is unable to find the order of $\underset{\sim}{G}(2,3,-2)$ directly but it does give that the index of a is 252. It also gives that the index of the subgroup $\langle a,b^6 \rangle$ is 252 and so b^6 is in the subgroup generated by a. Therefore $[a,b^6] = 1$, and by symmetry $[b,a^6] = 1$. Also since $ab^3a^{-1}b^{-1}a^2b^{-1} = 1$ and $ba^3b^{-1}a^{-1}b^2a^{-1} = 1$

$$ab^3a^{-1}b^{-1}a^2b^{-1}ba^3b^{-1}a^{-1}b^2a^{-1} = 1,$$

that is

$$b^5a^{-1}b^{-1}a^5b^{-1}a^{-1} = 1.$$

But $[a,b^6] = 1$ and $[b,a^6] = 1$ so

$$(ab)^3 = b^6a^6 = a^6b^6 = (ba)^3.$$

If we add on the relations that we have deduced hold in $\underset{\sim}{G}(2,3,-2)$, namely $[a,b^6] = 1$, $[b,a^6] = 1$, $(ab)^3 = (ba)^3$ then with this information the coset enumeration programme (1) now gives $|\underset{\sim}{G}(2,3,-2)| = 1512$.

We may also show that $b^{-1}a^3b^4a^{-1}b$, a^3 and $ba^2b^3a^{-2}b^{-1}$ are each words of order two, that any two of them generate a subgroup of order 4 and that the three of them generate the Sylow 2 - subgroup. By adding on the relations $[b^{-1}a^3b^4a^{-1}b,a^3] = 1$, $[a^3,ba^2b^3a^{-2}b^{-1}] = 1$, and $[b^{-1}a^3b^4a^{-1}b,ba^2b^3a^{-2}b^{-1}] = 1$ to the presentation for $\underset{\sim}{G}(2,3,-2)$ we show, since the resulting group still has order 1512 that the three words of order 2, $b^{-1}a^3b^4a^{-1}b, a^3$ and $ba^2b^3a^{-2}b^{-1}$ commute. Therefore the Sylow 2 - subgroup is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Further let H be the subgroup generated by ab^{-1} and $b^{-1}a$. Then from the modified Todd-Coxeter algorithm H is a normal subgroup of index 3 in $\underset{\sim}{G}(2,3,-2)$ and with $x = ab^{-1}$ and $y = b^{-1}a$,

$$\begin{aligned} H \cong \langle x,y \mid\ & x^3yx^{-1}y^{-1}x^{-2}y^{-1}xyxy^{-1}xyx^{-1}y^{-1}x^{-1}y = 1, \\ & x^3yx^{-1}yx^{-2}y^{-1}xy^{-1}x^{-1}y = 1, \\ & x^3yx^{-1}y^{-1}x^{-1}yxy^{-1}xyxy^{-1}x^{-2}y^{-1}x^{-1}y = 1, \\ & x^3yx^{-1}y^{-1}xy^{-1}x^{-2}yx^{-1}y = 1\rangle. \end{aligned}$$

If we abelianise these relations we get the group with presentation

$$\langle x,y \mid x = y = 1\rangle.$$

Thus H is perfect and since it has order 504 it is either PSL(2,8) the simple group of order 504 or has PSL(2,7) the simple group of order 168 as a homomorphic image. However we have shown that $\underset{\sim}{G}(2,3,-2)$ has a Sylow 2 - subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and therefore the second case is ruled out. Thus $H \cong PSL(2,8)$ and $\underset{\sim}{G}(2,3,-2)$ is an extension of $PSL(2,8)$ by $\mathbb{Z}_3$. $\underset{\sim}{G}(2,3,-2)$ is not soluble having a simple subgroup and is therefore neither metacyclic nor nilpotent. Further, it is not isomorphic to either of the two non-metacyclic groups of order 1512 discussed in section 4.3, namely $\underset{\sim}{F}(3,6)$ and $\underset{\sim}{F}(3,6,5,2)$. It does however seem likely that it is isomorphic to Edge's group of order 1512 (26) which also has $PSL(2,8)$ as a normal subgroup of index three, but we have so far failed to show that the $\mathbb{Z}_3$ action is identical in both cases.

3.4.7. The group $\underset{\sim}{G}(-2,2,-1)$.

$$\underset{\sim}{G}(-2,2,-1) \cong \langle a,b \mid ab^2a^{-1}b^{-1}ab^3 = 1,\ ba^2b^{-1}a^{-1}ba^3 = 1 \rangle.$$

Coset enumeration gives $|\underset{\sim}{G}(-2,2,-1)| = 1320 = 2^3.3.5.11$. The element a has order 30 and so a^{10} is an element of order 3. If we add on the relation $a^{10} = 1$ to the relations for $\underset{\sim}{G}(-2,2,-1)$ then we have a group of order 55 and so, since this order is not 440, the Sylow 3 - subgroups are not normal. Similarly if we add on the relation $a^6 = 1$, a^6 being an element of order 5, we find that this time the new group has order 24 and so the Sylow 5 - subgroups are not normal. The element $(ab^{-1})^6$ is of order 11 and adding on the relation $(ab^{-1})^6 = 1$ we obtain a group of order 120 and so there is a unique normal Sylow 11 - subgroup.

$(a^{-1}b^{-1}ab)^{11}$ and $(ab)^5$ are two elements of order 4 and together they generate the Sylow 2 - subgroup. By adding on the relation $(ab)^5 = 1$ to those of $\underset{\sim}{G}(-2,2,-1)$ we obtain a group of order 165.

Therefore, since the normal closure of $(ab)^5$ is of order 8, the Sylow 2 - subgroup is normal. Since it is generated by two elements of order 4, the Sylow 2 - subgroup is the quaternion group (since it is not abelian).

$G(-2,2,-1)$ is not nilpotent since it is not the direct product of its Sylow subgroups. However, it is soluble since the derived group, being of order $2^3.11$, is of the form $p^\alpha q^\beta$ where p and q are primes.

3.4.8. The group $G(-2,3,-1)$.

$$G(-2,3,-1) \cong \langle\, a,b \mid ab^3a^{-1}b^{-1}ab^3 = 1,\ ba^3b^{-1}a^{-1}ba^3 = 1 \,\rangle.$$

Coset enumeration gives $|G(-2,3,-1)| = 5832 = 2^3.3^6$, this being one of the largest orders obtained directly by the coset enumeration programme (1). The element a has order 24 and so there is a cyclic Sylow 2 - subgroup of order 8 generated by a^3. Adding the relation $a^3 = 1$ to those for $G(-2,3,1)$ we obtain a group of order 3 and so the Sylow 2 - subgroup is not normal. Since $|G(-2,3,-1):G'(-2,3,-1)| = 24$, $|G'(-2,3,-1)| = 3^5$ and therefore there is a unique Sylow 3 - subgroup. $aba^{-1}b^{-1}$ and $(ab^{-1})^4$ are two elements of order 9. The derived group may be generated by $(aba^{-1}b^{-1})$, $(ab)^2$ and $(ab^{-1})^4$. When we add to the presentation for $G(-2,3,-1)$ the relation $[aba^{-1}b^{-1},(ab^{-1})^4] = 1$ we obtain a group of order 1944 and since two elements in the Sylow 3 - subgroup do not commute, the Sylow 3 - subgroup is not abelian. ~~Hence~~ In fact $G(-2,3,-1)$ is not metabelian. Being of the form $p^\alpha q^\beta$ where p and q are primes $G(-2,3,-1)$ is soluble. Since the Sylow 2 - subgroup is not normal it is, however, not nilpotent.

3.4.9. Other groups $\underset{\sim}{G}(1,m,n)$.

We next consider other examples of the groups $\underset{\sim}{G}(1,m,n)$ where the orders of the groups have been determined by the coset enumeration programme (1) and where some other information is known about the groups.

From Theorem 3.2.7 we know that the three groups $\underset{\sim}{G}(3,3,1)$, $\underset{\sim}{G}(-1,-1,-1)$ and $\underset{\sim}{G}(-2,-2,1)$ have $SL(2,5)$ as a homomorphic image. Coset enumeration gives that each has order 120 and so each is isomorphic to $SL(2,5)$. (The group $\underset{\sim}{G}(3,3,1)$ is also discussed in (8) and (19), see 3.1.)

We next show that $\underset{\sim}{G}(2,2,2)$ and $\underset{\sim}{G}(2,2,-2)$ are isomorphic to D_∞ and that $\underset{\sim}{G}(2,2,4)$ is an extension of a group of order 32 by D_∞. By Theorem 3.3.7 we know that $\langle a^2,b^2 \rangle$ is a normal subgroup of $\underset{\sim}{G}(2,2,2n)$ and that $\underset{\sim}{G}(2,2,2n)/\langle a^2,b^2 \rangle$ is isomorphic to D_∞. We therefore have to show that for $\underset{\sim}{G}(2,2,2)$ and $\underset{\sim}{G}(2,2,-2)$ $a^2 = b^2 = 1$.

$$\underset{\sim}{G}(2,2,2) \cong \langle a,b \mid ab^2a^{-1}b^{-1}a^{-2}b^{-1} = 1, ba^2b^{-1}a^{-1}b^{-2}a^{-1} = 1 \rangle.$$

We have $ab^2a^{-1} = ba^2b^{-1}b^2 = ab^2ab^2$ and so $a^2 = b^{-2}$. Since $a^2 = b^{-2}$ the relation $ab^2a^{-1}b^{-1}a^{-2}b^{-1} = 1$ then gives $b^2 = 1$ and so $a^2 = b^2 = 1$. Therefore $\underset{\sim}{G}(2,2,2) \cong D_\infty$.

$\underline{\underset{\sim}{G}(2,2,-2).}$ By Theorem 3.4.1 we can obtain a presentation for $H = \langle a,b^2 \rangle$. Now

$$H \cong \langle x,y \mid R(k) = 1,\ k = 1,2,3, \dots \rangle$$

where the relations $R(k)$ are as defined in Theorem 3.4.1. The first four of these relations are

$$x^{-2}yx^{-1}yxy^{-1} = 1, \qquad (1)$$

$$x^2y^{-1}xy^{-1}xyx^{-2}y^{-1} = 1, \qquad (2)$$

$$(yx^{-1})^4y^{-1}xy^{-1}x^{-1} = 1, \qquad (3)$$

$$xy^{-1}x^3y^{-1}xy^{-1}x^{-2}yxy^{-1} = 1, \qquad (4)$$

From relations (1) and (4)

$$xy^{-1}x^3y^{-2}x^2y^{-1} = 1.$$

But (2) gives that

$$y^{-1}x^2y^{-1}xy^{-1} = x^2y^{-1}x^{-1},$$

and so

$$x^3y^{-1}x^2y^{-1}x^{-1} = 1,$$

that is

$$(x^2y^{-1})^2 = 1. \tag{5}$$

From (2) and (5) we deduce that

$$y^{-1}xy = x^3. \tag{6}$$

From (3) and (6),

$$(yx^{-1})^3y^{-1}xx^{-3}x^{-1} = 1,$$

that is

$$x^{-1}x^{-2}(yx^{-1})^3y^{-1} = 1,$$

that is, using (5),

$$x^{-1}y^{-1}xyx^{-1}yx^{-1}y^{-1} = 1.$$

This reduces using (6) so that

$$xyx^{-1}y^{-1} = 1, \tag{7}$$

and hence

$$x^2 = 1. \tag{8}$$

From (1) we now deduce that $y = 1$. Equations (2), (3) and (4) are satisfied by $x^2 = 1$ and $y = 1$. Further, since the relations $R(k) = 1$ for $k \geq 5$ hold in the group $\langle x,y \mid y = 1,\ x^2 = 1\rangle$, H is isomorphic to the cyclic group of order 2. This gives $a^2 = 1$ and also $b^2 = 1$. Thus by Theorem 3.3.7 $\underset{\sim}{G}(2,2,-2)$ is isomorphic to D_∞, the infinite dihedral group.

$\underset{\sim}{G}(2,2,4)$. By Theorem 3.4.1 we again obtain a presentation for $H = \langle a,b^2 \rangle$. We show first that

$$H \cong \langle x,y \mid R(k) = 1,\ k = 1,2,3,4,5\rangle.$$

For this we have to show that H has only five relations. In Theorem 3.4.1 we have n = 2 and thus

$$w(k) = w(k-2)[w(k-3)]^2, \qquad k \geq 5,$$

and

$$[w(k+2)]^2 w(k+1)w(k)[w(k-1)]^{-1}[w(k)]^{-1} = 1, \qquad k \geq 3.$$

Therefore, since $w(k+2) = w(k)[w(k-1)]^2$, $k \geq 3$,

$$w(k+2)w(k+1)w(k)w(k-1) = 1, \qquad k \geq 3.$$

Then

$$w(k+3)w(k+2)w(k+1)w(k) = 1,$$

and so

$$w(k+3) = w(k-1), \qquad k \geq 3.$$

Therefore H has only five relations. Hence

$$H \cong \langle x,y \mid R(1) = R(2) = R(3) = R(4) = R(5) = 1\rangle,$$

where the relations are

$$(xy^2x)^2yxy^{-1}x^{-1} = 1, \tag{1}$$

$$(yx^4)^2xy^2xyx^{-2}y^{-1} = 1, \tag{2}$$

$$(xy^2)^4yx^5y^2xy^{-1}x^{-1}y^{-2}x^{-1} = 1, \tag{3}$$

$$(yx^4(xy^2x)^2)^2(xy^2)^2yx^3y^{-2}x^{-5}y^{-1} = 1, \tag{4}$$

$$((xy^2)^2(yx^4)^2)^2yx^4(xy^2x)^2xy^2xy^2x^{-4}y^{-1}(xy^2)^{-2} = 1. \tag{5}$$

We now simplify these relations to find a more manageable presentation for H. From (1) and (2)

$$yx^2y^2xyx = 1, \tag{6}$$

and

$$x^2yx^5y^2xy = 1. \tag{7}$$

Using (6), (7) may be rewritten as

$$y^{-1}xy = x^{-3}, \tag{8}$$

and then (6) becomes

$$yx^3x^{-1}y^2xyx = 1,$$

that is

$$yx^{-1}y^2xy = 1,$$

or

$$xy^2x^{-1} = y^{-2}. \tag{9}$$

Equation (3) is $yxy^2xy^3x^5y^2x = 1$ and from (9) this may be rewritten as

$$yy^{-2}xxy^3x^5xy^{-2} = 1$$

that is

$$y^{-3}x^2y^3x^6 = 1,$$

or

$$y^{-1}x^2y = x^{-6},$$

but this relation is implied by (8).

Equation (4) is $(yx^5y^2x^2y^2x)^2(xy^2)^2yx^3y^{-2}x^{-5}y^{-1} = 1.$ From (9) $(xy^2)^2 = x^2$ and so $x^2y^2xyx^5y^2x^2y^2x^3yx^3 = 1$, that is, using (9),

$$x^2xy^{-1}x^5x^2y^4x^3yx^3 = 1.$$

From (8) this may be rewritten as

$$y^{-1}x^{-1}x^7y^3x^{-1}yx^{-1}y = 1,$$

or

$$x^5y^3x^{-1}y = 1.$$

Finally, using (9), this becomes

$$yx^{-1}y^{-1} = x^{-5}. \tag{10}$$

Equation (5) is $yx^4(xy^2)^2(yx^4)^3(xy^2x)^2(xy^2)^2 = 1$. From (8) and (9) $(xy^2)^2 = x^2$ and $yx^4 = x^{-1}yx$ and so (5) becomes

$$x^{-1}yxx^2x^{-1}y^3xxy^2xxy^2xx^2 = 1,$$

that is

$$yx^2y^3x^2y^2x^2y^2x^2 = 1. \qquad (11)$$

But $xyx^2y^2 = y^{-1}x^{-1}$ from (6) and so (11) becomes

$$y^{-1}x^{-1}yx^2y^2x^2y^2x = 1,$$

that is

$$y^{-2}yx^{-1}y^{-1}y^2x^2y^2x^2y^2x = 1,$$

and from (9) and (10) this becomes $y^8 = 1$, which we shall show is also a consequence of (8), (9) and (10).

Thus the presentation for H simplifies to

$$H \cong \langle x,y \mid y^{-1}xy = x^{-3},\ xy^2x^{-1} = y^{-2},\ yx^{-1}y^{-1} = x^{-5} \rangle$$

and so H is a group of order 64. (From these relations we can deduce that $x^{16} = 1$ and $y^4 = x^8$ and therefore as required $y^8 = 1$. For $y^{-1}xy = x^{-3}$ gives $yx^{-3}y^{-1} = x$ and $yx^{-1}y^{-1} = x^{-5}$ gives $yx^{-3}y^{-1} = x^{-15}$ from which we obtain the result $x^{16} = 1$. Also from $y^{-1}xy = x^{-3}$ and $yx^{-1}y^{-1} = x^{-5}$, $y^{-1}xy^2x^{-1}y^{-1} = x^{-8}$. But $xy^2 = y^{-2}x^{+1}$ from which the result $y^4 = x^8$ follows.) $\underset{\sim}{G}(2,2,4)$ is therefore an extension of a group of order 32 by D_∞ since $\langle a^2, b^2\rangle$ has index 2 in $\langle a, b^2\rangle$.

We next show that the groups $\underset{\sim}{G}(4,-1,-1)$ and $\underset{\sim}{G}(4,-3,1)$ are infinite.

Coset enumeration gives $|\underset{\sim}{G}(4,-1,-1) : \langle ab,ba\rangle| = 24$. Using the Wilde programme (47) and the programme CCRG, see 2.5, we find that if $x = ab$ and $y = ba$ then the subgroup $\langle x,y \rangle$ is the free group on x and y. Therefore $\underset{\sim}{G}(4,-1,-1)$ is infinite.

Next for $G(4,-3,+1)$ the subgroup $H = \langle a^2, b\rangle$ has index 3. The programmes (47) and CCRGAB, see 2.5, then give

$$H/H' \cong \langle x,y \mid x^{16} = y^{16}, [x,y] = 1\rangle$$

where $x = a^2$ and $y = b$. This implies that H is infinite and therefore that $G(4,-3,1)$ is infinite.

$G(4,1,2)$ is a group of order 55. Since $|G(4,1,2):G'(4,1,2)| = 5$, $|G'(4,1,2)| = 11$ and so $G(4,1,2)$ is metacyclic.

$G(4,-1,1)$ is a group of order 216. Since $|G(4,-1,1):G'(4,-1,1)| = 24$, $G'(4,-1,1) = 9$ and so $G(4,-1,1)$ is metabelian.

$G(4,-1,-2)$ is a cyclic group of order 21.

$G(4,-2,1)$ is a group of order 1015. Since $|G(4,-2,1):G'(4,-2,1)| = 35$, $G'(4,-2,1) = 29$ and so $G(4,-2,1)$ is metabelian.

$G(4,-2,-1)$ is a group of order 385. Since $|G(4,-2,-1):G'(4,-2,-1)| = 35$, $|G'(4,-2,-1)| = 11$ and so $G(4,-2,-1)$ is metabelian.

$G(4,-3,-1)$ is a group of order 432. Since $|G(4,-3,-1):G'(4,-3,-1)| = 48$, $|G'(4,-3,-1)| = 9$ and so $G(4,-3,-1)$ is metabelian.

$G(3,-1,3)$ is a group of order 497. Since $|G(3,-1,3):G'(3,-1,3)| = 7$, $|G'(3,-1,3)| = 71$ and so $G(3,-1,3)$ is metacyclic.

$G(2,3,-3)$ is a group of order 24. This order cannot be determined directly by the coset enumeration programme (1) but it does show that a has index 3 in $\underset{\sim}{G}(2,3,-3)$ as also does the subgroup $\langle a,b^2\rangle$. As described in 2.6 we may add the relation $[a,b^2] = 1$, and with this extra relation the order of $\underset{\sim}{G}(2,3,-3)$ is found to be 24. $\underset{\sim}{G}(2,3,-3)/\underset{\sim}{G}'(2,3,-3)$ is isomorphic to $\mathbb{Z}_8$ and so $\underset{\sim}{G}(2,3,-3)$ is ~~the~~ a binary polyhedral group $\langle -2,2,3\rangle$, see 1.4.

$\underset{\sim}{G}(2,2,5)$ is a metacyclic group of order 125.

$\underset{\sim}{G}(2,2,5) \cong \langle a,b \mid ab^2a^{-1}b^{-1}a^{-5}b^{-1} = 1,\ ba^2b^{-1}a^{-1}b^{-5}a^{-1} = 1\rangle.$

The subgroup $\langle a\rangle$ is normal in $\underset{\sim}{G}(2,2,5)$ and $\underset{\sim}{G}(2,2,5)/G'(2,2,5) \cong \mathbb{Z}_5\times\mathbb{Z}_5$. Hence $\langle a^5\rangle \cong \langle b^5\rangle \cong \underset{\sim}{G}'(2,2,5)$. Since a^5 and b^5 belong to $Z(\underset{\sim}{G}(2,2,5))$, the centre of $\underset{\sim}{G}(2,2,5)$, $b^{-2}ab^2 = a^6$ and $a^{-2}ba^2 = b^6$. Then $a^{-2}b^5a^2 = b^{30}$ and so $b^{25} = 1$. Also $b^{-1}a^{-2}b = b^5a^{-2}$ gives $b^{-1}a^{-4}b = b^{10}a^{-4}$ and so $b^{-1}ab = b^{10}a$. Thus $b^{-2}ab^2 = b^{20}a$ and this together with $b^{-2}ab^2 = a^6$ gives $a^5 = b^{-5}$.

$\underset{\sim}{G}(-1,2,1)$. The subgroup generated by $\langle ab,ba\rangle$ can be shown by the programme (1) to have index 4 in $\underset{\sim}{G}(-1,2,1)$. With $x = ab$ and $y = ba$, the Wilde programme (47) and CCRG, see 2.5, gives as a presentation for H the group generated by x and y subject to the six relations

$$xy^{-1}x^{-2}y^{-2}x^{-1}yx^2y^2 = 1, \tag{1}$$

$$xy^{-1}xy^2x^2yx^{-1}y = 1, \tag{2}$$

$$xy^2x^2yx^{-1}yx^2y^3x^{-1}y^2 = 1, \tag{3}$$

$$y^3x^2yx^{-1}yx^2yx^{-3}yxy^2x^2 = 1, \tag{4}$$

$$x^2y^{-2}x^{-2}y^{-1}xy^{-1}x^{-2}y^{-3}x^{-2}y^{-2} = 1, \tag{5}$$

$$y^{-3}x^{-2}y^3x^2 = 1. \tag{6}$$

From (2),

$$xy^2x^2yx^{-1} = yx^{-1}y^{-1}, \tag{7}$$

and substituting in (1) we obtain

$$yxy^{-1}yx^2y^2 = 1,$$

that is

$$x^3y^3 = 1. \tag{8}$$

From (3) and (7) $yx^{-1}y^{-1}yx^2y^3x^{-1}y^2 = 1$ and since by (8) y^3 commutes with x,

$$x^6 = 1. \tag{9}$$

Equation (1) may be rewritten as

$$(xy^{-1})^2(x^{-1}y)^2 = 1. \tag{10}$$

It is then straightforward to check that (4), (5) and (6) are consequences of (8), (9) and (10). Thus

$$H = \langle x,y \mid x^6 = 1,\ x^3y^3 = 1,\ (xy^{-1})^2(x^{-1}y)^2 = 1\rangle.$$

Coset enumeration now shows that H is of order 144 and therefore that $\underset{\sim}{G}(-1,2,1)$ is a group of order 576.

$\underset{\sim}{G}(0,3,-2)$ is a group of order 600. $|\underset{\sim}{G}(0,3,-2):\underset{\sim}{G}'(0,3,-2)| = 5$ and in fact $\underset{\sim}{G}'(0,3,-2)$ may be generated by $[a^{-1},b^{-1}]$ and $[a^{-1},b]$. Since we may show that these commutators commute $\underset{\sim}{G}(0,3,-2)$ is metabelian.

$\underset{\sim}{G}(-2,1,-1)$ is a group of order 120. Since $|\underset{\sim}{G}(-2,1,-1):\underset{\sim}{G}'(-2,1,-1)| = 8$, $|\underset{\sim}{G}'(-2,1,-1)| = 15$ and so $\underset{\sim}{G}(-2,1,-1)$ is metabelian.

3.4.10. Computer results.

We now note the other groups which have had their orders determined by the coset enumeration programme (1) and which are entered in the tables in subsection 3.4.11.

	Order		Order		Order
$\underset{\sim}{G}(3,2,2)$	24;	$\underset{\sim}{G}(3,-1,-1)$	120;	$\underset{\sim}{G}(3,-1,-3)$	7
$\underset{\sim}{G}(3,-2,2)$	609;	$\underset{\sim}{G}(3,-2,1)$	165;	$\underset{\sim}{G}(3,-3,1)$	1015
$\underset{\sim}{G}(3,-3,-1)$	385;	$\underset{\sim}{G}(3,-3,-2)$	2048;	$\underset{\sim}{G}(3,-4,1)$	1200
$\underset{\sim}{G}(2,5,1)$	24;	$\underset{\sim}{G}(2,4,1)$	3;	$\underset{\sim}{G}(2,3,2)$	3
$\underset{\sim}{G}(2,2,1)$	1;	$\underset{\sim}{G}(2,-1,5)$	1200;	$\underset{\sim}{G}(2,-1,4)$	301
$\underset{\sim}{G}(2,-1,2)$	55;	$\underset{\sim}{G}(2,-1,1)$	1;	$\underset{\sim}{G}(2,-1,-2)$	5
$\underset{\sim}{G}(2,-1,-4)$	7;	$\underset{\sim}{G}(2,-1,-5)$	48;	$\underset{\sim}{G}(2,-2,3)$	497
$\underset{\sim}{G}(2,-2,1)$	165;	$\underset{\sim}{G}(2,-2,-3)$	7;	$\underset{\sim}{G}(2,-3,3)$	240
$\underset{\sim}{G}(2,-3,2)$	1344;	$\underset{\sim}{G}(2,-3,+1)$	216;	$\underset{\sim}{G}(2,-3,-3)$	48
$\underset{\sim}{G}(2,-4,1)$	280;	$\underset{\sim}{G}(1,5,3)$	7;	$\underset{\sim}{G}(1,5,-1)$	120
$\underset{\sim}{G}(1,5,-2)$	24;	$\underset{\sim}{G}(1,5,-3)$	56;	$\underset{\sim}{G}(1,4,5)$	16
$\underset{\sim}{G}(1,4,4)$	7;	$\underset{\sim}{G}(1,4,-1)$	8;	$\underset{\sim}{G}(1,4,-2)$	5
$\underset{\sim}{G}(1,3,4)$	24;	$\underset{\sim}{G}(1,3,3)$	5;	$\underset{\sim}{G}(1,3,-1)$	3
$\underset{\sim}{G}(1,2,5)$	192;	$\underset{\sim}{G}(1,2,4)$	960;	$\underset{\sim}{G}(1,2,3)$	8
$\underset{\sim}{G}(1,2,2)$	24;	$\underset{\sim}{G}(1,-1,5)$	168;	$\underset{\sim}{G}(1,-1,4)$	24
$\underset{\sim}{G}(1,-1,3)$	5;	$\underset{\sim}{G}(1,-2,5)$	144;	$\underset{\sim}{G}(1,-2,4)$	56
$\underset{\sim}{G}(1,-3,5)$	171;	$\underset{\sim}{G}(0,3,2)$	5;	$\underset{\sim}{G}(-1,5,2)$	2048
$\underset{\sim}{G}(-1,3,-1)$	165;	$\underset{\sim}{G}(-1,2,-1)$	40;	$\underset{\sim}{G}(-1,2,-2)$	5
$\underset{\sim}{G}(-1,1,-3)$	55;	$\underset{\sim}{G}(-1,-1,5)$	125;	$\underset{\sim}{G}(-1,-1,3)$	27
$\underset{\sim}{G}(-1,-1,-2)$	8;	$\underset{\sim}{G}(-1,-1,-3)$	27;	$\underset{\sim}{G}(-1,-3,1)$	3
$\underset{\sim}{G}(-1,-3,-1)$	3;	$\underset{\sim}{G}(-2,4,-1)$	1015;	$\underset{\sim}{G}(-2,-1,2)$	648
$\underset{\sim}{G}(-2,-1,-2)$	3;	$\underset{\sim}{G}(-2,-2,-1)$	1.		

3.4.11. Tables.

We now give tables showing, where known, the orders of the groups $\underset{\sim}{G}(l,m,n)$, $-2 \leq l \leq 5$; $-4 \leq m,n \leq 5$. If there are two entries in a box then the upper half of the box gives the order of the group and the lower half one of the theorems or lemmas, for there may be several, from which it has been deduced. (Every theorem or lemma comes from chapter 3 so we omit the initial 3 in the reference.)

$\underset{\sim}{G}(5,m,n)$

n \ m	−4	−3	−2	−1	0	1	2	3	4	5
5					∞ 3.3					∞ 3.6
4					9 3.3	∞ 2.6				
3					16 3.3		∞ 2.6			
2					21 3.3			∞ 2.6		
1			1200	280	24 3.3	1800 2.4			∞ 2.6	
0	729 3.9	512 3.9	343 3.9	216 3.9	∞ 3.9	64 3.9	27 3.9	8 3.9	1 3.9	∞ 3.9
−1		1827		4200	24 3.3	105 2.4			∞ 2.6	
−2				∞ 3.4	21 3.3	24 2.4		∞ 2.6		
−3					16 3.3	7 2.4	∞ 2.6			
−4		192	231	40	9 3.3	∞ 2.6	7 3.2	24	105	64 3.8

$\underset{\sim}{G}(4,m,n)$

n \ m	-4	-3	-2	-1	0	1	2	3	4	5
5				∞ 2.6	9 3.3					
4	∞ 3.6		∞ 3.6		∞ 3.6		∞ 3.6		∞ 3.6	
3					7 3.3	∞ 2.6				
2	∞ 3.6		∞ 3.6		∞ 3.6	55	∞ 3.6		∞ 3.6	
1		∞	1015	216	15 3.3	192 2.4		∞ 2.6		∞ 2.6
0	∞ 3.6	343 3.9	∞ 3.6	125 3.9	∞ 3.6	27 3.9	∞ 3.6	1 3.9	∞ 3.6	1 3.9
-1		432	385	∞	15 3.3	24 2.4		∞ 2.6		∞ 2.6
-2	∞ 3.6		∞ 3.6	21	∞ 3.6	5 2.4	∞ 3.6		∞ 3.6	
-3		960 3.16	81 3.15	48 3.16	7 3.3	∞ 2.6	5 3.2	24 3.16	27 3.8	192 3.16
-4	∞ 3.6		∞ 3.6		∞ 3.6	7 2.4	∞ 3.6		∞ 3.6	

$\underset{\sim}{G}(3,m,n)$

n \ m	-4	-3	-2	-1	0	1	2	3	4	5
5			∞ 2.6		16 3.3					
4				∞ 2.6	7 3.3					
3		∞ 3.6		497	∞ 3.3			∞ 3.6		
2			609		5 3.3	∞ 2.6	24			∞ 2.6
1	1200	1015		165	8 3.3	24 2.4	∞ 2.6	120	∞ 2.6	
0	343 3.9	∞ 3.9	125 3.9	64 3.9	∞ 3.9	8 3.9	1 3.9	∞ 3.9	1 3.9	8 3.9
-1		385		120	8 3.3	3 2.4	∞ 2.6	1	∞ 2.6	
-2		2048	168 3.16	24 3.15	5 3.3	∞ 2.6	3 3.2	8 3.15	24 3.16	∞ 2.6
-3		∞ 3.6		7	∞ 3.3	5 2.4		∞ 3.6		
-4				∞ 2.6	7 3.3	24 2.4				

$\underset{\sim}{G}(2,m,n)$

n \ m	−4	−3	−2	−1	0	1	2	3	4	5
5		∞ 2.6		1200	21 3.3	1512 2.4	125			
4	∞ 3.6		∞ 3.6	301	∞ 3.3	465 2.4	∞ 3.6		∞ 3.6	
3		240	497	∞ 2.6	5 3.3	120 2.4	884,736			∞ 2.6
2	∞ 3.6	1344	∞ 3.6	55	∞ 3.3	21 2.6	∞ 3.6	3	∞ 3.6	
1	280	216	165	24	3 3.3	∞ 2.6	1	∞ 2.6	3	24
0	∞ 3.6	125 3.9	∞ 3.6	27 3.9	∞ 3.3	1 3.6	∞ 3.9	1 3.6	∞ 3.9	27 3.6
−1	1085 3.16	360 3.16	105 3.16	24 3.16	3 3.3	∞ 2.6	1 3.2	∞ 2.6	21 3.16	120 3.16
−2	∞ 3.6		∞ 3.6	5	∞ 3.3	3 2.4	∞ 3.6	1512	∞ 3.6	
−3		48	7	∞ 2.6	5 3.3	24 2.4	6912	24		∞ 2.6
−4	∞ 3.6		∞ 3.6	7	∞ 3.3	105 2.4	∞ 3.6		∞ 3.6	

$\underset{\sim}{G}(1,m,n)$

n \ m	-4	-3	-2	-1	0	1	2	3	4	5
5	∞ 2.6	171	144	168	24 2.3	125 3.9	192		16	
4	171 2.3	∞ 2.6	56	24	15 2.3	64 3.9	960	24	7	∞ 2.6
3	144 2.3	56 2.3	∞ 2.6	5	8 2.3	27 3.9	8	5	∞ 2.6	7
2	168 2.3	24 2.3	5 2.3	∞ 2.6	3 2.3	8 3.9	24	∞ 3.13	∞ 3.13	∞ 3.13
1	24 2.3	15 2.3	8 2.3	3 2.3	∞ 3.1	1 3.1	∞ 3.1	3 3.1	8 3.1	15 3.1
0	125 2.3	64 2.3	27 2.3	8 2.3	1 2.3	∞ 3.9	1 3.9	8 3.9	27 3.9	64 3.9
-1	192 2.3	960 2.3	8 2.3	24 2.3	∞ 2.3	1 3.5	∞ 2.6	3	8	120
-2		24 2.3	5 2.3	∞ 2.3	3 2.3	8 2.3	3 2.3	∞ 2.6	5	24
-3	16 2.3	7 2.3	∞ 2.3	∞ 2.3	8 2.3	27 2.3	8 2.3	5 2.3	∞ 2.6	56
-4		∞ 2.3	7 2.3	∞ 2.3	15 2.3	64 2.3	120 2.3	24 2.3	56 2.3	∞ 2.6

$\underset{\sim}{G}(0,m,n)$

n \ m	-4	-3	-2	-1	0	1	2	3	4	5
5				∞ 2.2	∞ 2.2	576 2.2				∞ 3.6
4	∞ 2.2	192 2.2	∞ 2.2	1800 2.2	∞ 2.2	120 2.2	∞ 2.2		∞ 3.6	
3	192 2.2	∞ 2.2	600 2.2	192 2.2	∞ 2.2	24 2.2	5 2.2	∞ 3.6		
2	∞ 2.2	600 2.2	∞ 2.2	24 2.2	∞ 2.2	3 2.2	∞ 3.6	5	∞ 3.6	
1	1800 2.2	192 2.2	24 2.2	∞ 2.2	1 2.2	∞ 3.6	3 3.2	24 3.16	120 3.16	576 3.16
0	∞ 2.2	∞ 2.2	∞ 2.2	1 2.2	∞ 3.9	1 3.9	∞ 3.9	∞ 3.9	∞ 3.9	∞ 3.9
-1	120 2.2	24 2.2	3 2.2	∞ 2.2	1 2.6	∞ 2.6	24 3.16	192 2.4	1800 3.16	∞ 3.16
-2	∞ 2.2	5 2.2	∞ 2.2	3 2.2	∞ 2.2	24 2.2	∞ 3.6	600	∞ 3.6	
-3		∞ 2.2	5 2.2	24 2.2	∞ 2.2	192 2.2	600 2.2	∞ 3.6	~~192~~ ~~2.4~~	
-4	∞ 2.2		∞ 2.2	120 2.2	∞ 2.2	1800 2.2	∞ 2.2	192 2.2	∞ 3.6	

$\underset{\sim}{G}(-1,m,n)$

n \ m	-4	-3	-2	-1	0	1	2	3	4	5
5				125	24 3.3	147 2.4			∞ 2.6	
4				7680	15 3.3	24 2.4		∞ 2.6		
3	∞ 2.6			27	8 3.3	5 2.4	∞ 2.6			
2	20	∞ 2.6	21 3.16	8 3.8	3 3.3	∞ 2.6	5 3.2	24 3.15	147 3.16	2048
1		3	∞ 2.6	1	∞ 3.3	3 2.4	576			
0	27 3.9	8 3.9	1 3.9	∞ 2.6	1 3.3	8 3.9	27 3.9	64 3.9	125 3.9	216 3.9
-1		3	∞ 2.6	120	∞ 3.3	21 2.4	40	165.		
-2		∞ 2.6	3	8	3 3.3	∞ 2.6	5			
-3	∞ 2.6			27	8 3.3	55	∞ 2.6			
-4					15 3.3			∞ 2.6		

$\underset{\sim}{G}(-2,m,n)$

n \ m	-4	-3	-2	-1	0	1	2	3	4	5
5					21 3.3	48 2.4		∞ 2.6		
4	∞ 3.6		∞ 3.6		∞ 3.6	7 2.4	∞ 3.6		∞ 3.6	
3		120 3.16	27 3.8	24 3.16	5 3.3	∞ 2.6	7 3.2	48 3.16	81 3.15	600 3.16
2	∞ 3.6		∞ 3.6	648	∞ 3.6	5 2.4	∞ 3.6		∞ 3.6	
1		∞ 2.6	120	∞ 2.6	3 3.3	24 2.4				
0	∞ 3.6	1 3.9	∞ 2.6	1 3.9	∞ 3.6	27 3.9	∞ 3.6	125 3.9	∞ 3.6	343 3.9
-1		∞ 2.6	1	∞ 2.6	3 2.1	120 2.4	1320	5832.	1015	
-2	∞ 3.6		∞ 3.6	3	∞ 2.6		∞ 3.6		∞ 3.6	
-3					5 3.3	∞ 2.6				
-4	∞ 3.6		∞ 3.6		∞ 3.3		∞ 2.6		∞ 3.6	

$\underset{\sim}{G}(1,1,n)$

n \ ℓ	−4	−3	−2	−1	0	1	2	3	4	5
5	∞	9	48	147	576	125	1512			
4	9	∞	7	24	120	64	465			∞
3	48	7	∞	5	24	27	120		∞	
2	147	24	5	∞	3	8	21	∞	55	
1	576	120	24	3	∞	1	∞	24	192	1800
0	125	64	27	8	1	∞	1	8	27	64
−1	1512	465	120	21	∞	1	∞	3	24	105
−2				∞	24	8	3	∞	5	24
−3			∞	55	192	27	24	5	∞	7
−4		∞			1800	64	105	24	7	∞

$\underset{\sim}{G}(1-n,m,n)$

n \ m	−4	−3	−2	−1	0	1	2	3	4	5
5	125	576	147	48	9	∞	11	72	273	1344
4	465	64	120	24	7	∞	9	40	264	192
3		120	27	24	5	∞	7	48	81	600
2	20	∞	21	8	3	∞	5	24	147	2048
1	1800	192	24	∞	1	∞	3	24	120	576
0	125	64	27	8	1	∞	1	8	27	64
−1	1085	360	105	24	3	∞	1	∞	21	120
−2		2048	168	24	5	∞	3	8	24	∞
−3		960	81	48	7	∞	5	24	27	192
−4		192	231	40	9	∞	7	24	105	64

3.4.12. Conclusion.

In conclusion we looked at, and failed to answer, the following problems. We may first ask whether all the groups $\underset{\sim}{G}(-1,-1,n)$ $n \neq 0$ are finite. However, despite much computing effort, we were unable to determine the order of $\underset{\sim}{G}(-1,-1,-4)$. A second question arising from the first is to ask when $\underset{\sim}{G}(-1,-1,n)$ is isomorphic to $\underset{\sim}{G}(1,1,n)$ and when $\underset{\sim}{G}(-1,-1,-n)$ is isomorphic to $\underset{\sim}{G}(-1,-1,n)$. We know by Theorem 3.2.7 that $\underset{\sim}{G}(-1,-1,n)$ has $SL(2,5)$ as a homomorphic image whenever $n \equiv 4 \bmod 5$. A consequence of this is that, since $\underset{\sim}{G}(-1,-1,1)$ and $\underset{\sim}{G}(1,1,-1)$ are both the trivial group, then, when $n = 1$, $\underset{\sim}{G}(-1,-1,-1)$ is neither isomorphic to $\underset{\sim}{G}(-1,-1,1)$ nor to $\underset{\sim}{G}(1,1,-1)$. Do the isomorphisms hold except when the conditions of Theorem 3.2.7 are satisfied, that is when $l = m = -1$ and $n \equiv 4 \bmod 5$? The final problem is of course to complete the tables and to look for further general results.

CHAPTER 4

FIBONACCI-TYPE GROUPS

4.1 Introduction

In 1908 G. A. Miller (39) in a paper in the American Journal of Mathematics entitled 'Groups generated by n operators each of which is the product of the n-1 remaining ones' discussed the groups indicated by the title. Interest in similar types of groups received impetus from the following problem posed by J.H. Conway (20) in 1965 in the American Mathematical Monthly.

"Show that the group G_5 generated by five generators a,b,c,d,e subject only to the relations ab = c, bc = d, cd = e, de = a, ea = b is cyclic of order 11."

Much interest was aroused by the Conway problem as evidenced by the number and variety of solutions, a solution of mine being one of them. These solutions appeared in the American Mathematical Monthly (21) in 1967. I solved the problem by using the Todd-Coxeter coset enumeration algorithm in the following way.

G_5 is cyclic of order 11 if we can show that $a^{11} = 1$, and also if we can show that b,c,d and e are each some power of a. The modified Todd-Coxeter algorithm, see 2.2, gives the following nine relations:

$$c = ab, \qquad (1)$$

$$d = bc, \qquad (2)$$

$$bce = de = a, \qquad (3)$$

$$bcb = bcea = a^2, \qquad (4)$$

$$bcd = bcbc = a^2c = a^3b, \qquad (5)$$

$$be = bcd = a^3b, \qquad (6)$$

$$bca = bcde = a^3be = a^6b, \tag{7}$$

$$e = ba^{-1} = a^{-6}bc, \tag{8}$$

$$bd = bc^{-1}e = a^{-1}e = a^{-7}bc, \tag{9}$$

Since $d = bc = a^7bd$,

$$a^7b = 1,$$

Also $bca = a^6b = a^{-1}$ and therefore

$$bc = a^{-2}.$$

The relation $ab = c$ now gives

$$aa^{-7} = a^7a^{-2},$$

that is

$$a^{11} = 1.$$

It follows immediately that $b = a^4$, $c = a^5$, $d = a^9$ and $e = a^3$.

An alternative solution based on a proof by J. A. Wenzel (21) is inserted at this stage because some of the ideas in the proof are in fact reiterated in some of our later work. From the given presentation

$$c = ab, \tag{1}$$

$$d = bc = bab, \tag{2}$$

$$e = cd = ab^2ab. \tag{3}$$

Thus G_5 has a presentation on the two generators a and b subject to the two relations

$$a = babab^2ab, \tag{4}$$

$$b = ab^2aba. \tag{5}$$

From relation (5) and using relation (4)

$$b = ab^2aba = ab^2ab^2abab^2ab = ab^5ab$$

Hence

$$b^5 = a^{-2}. \tag{6}$$

Also from relation (4)

$$a^2 = bab\,a\,b^2aba,$$

and so

$$b^{-5} = a^2 = bab^2,$$

that is

$$a = b^{-8}. \qquad (7)$$

From (6) and (7)

$$b^{11} = 1, \qquad (8)$$

and from (4) and (5) and using (6) and (7), $b^{11} = 1$ and $b^{22} = 1$. Thus G_5 is the cyclic group of order 11.

Several authors including Lyndon, Coxeter, Moser and Mendelsohn considered a generalisation G_n to n generators. G_1 and G_2 are easily seen to be trivial, G_3 is the quaternion group of order 8 and G_4 is $\mathbb{Z}_5$, the cyclic group of order 5. G_6 was shown to be infinite by myself, Coxeter and Mendelsohn. Mendelsohn claimed further that G_7 was infinite but Brunner (5) has since ~~shown~~ stated that this is false and that G_7 is in fact the cyclic group of order 29.

To show that G_6 was infinite, I used the modified Todd-Coxeter algorithm. G_6 has a presentation

$$\langle a,b,c,d,e,f \mid ab=c,\ bc=d,\ cd=e,\ de=f,\ ef=a,\ fa=b \rangle .$$

Let H be the subgroup $\langle f^2, b^2, ea^{-1}df^{-1} \rangle$ and let $h_1 = f^2$, $h_2 = b^2$, $h_3 = ea^{-1}df^{-1}$. Then we obtain the coset enumeration tables:

$a\ b\ c^{-1}$	$b\ c\ d^{-1}$	$c\ d\ e^{-1}$	$d\ e\ f^{-1}$	$e\ f\ a^{-1}$	$f\ a\ b^{-1}$
1 6 2 1	1 3 4 1	1 2 5 1	1 4 2 1	1 5 6 1	1 2 3 1
2 3 1 2	2 6 5 2	2 1 4 2	2 5 1 2	2 4 3 2	2 1 6 2
3 5 4 3	3 1 2 3	3 4 6 3	3 2 4 3	3 6 5 3	3 4 1 3
4 1 3 4	4 5 6 4	4 3 2 4	4 6 3 4	4 2 1 4	4 3 5 4
5 2 6 5	5 4 3 5	5 6 1 5	5 3 6 5	5 1 2 5	5 6 4 5
6 4 5 6	6 2 1 6	6 5 3 6	6 1 5 6	6 3 4 6	6 5 2 6

$$\begin{array}{ccc} f\ f & b\ b & e\ a^{-1}d\ f^{-1} \\ 1\ 2\ 1 & 1\ 3\ 1 & 1\ 5\ 3\ 2\ 1 \end{array}$$

where the coset representatives of the six cosets are 1[illegible]}, 2=f, 3=b, 4=d, 5=e and 6=a. The corresponding ~~regulations~~ relations between the coset representatives are tabulated below.

	a	b	c	d	e	f
1	6	3	$h_2^{-1}h_1^{-1}2$	4	5	2
2	3	$h_1 6$	$h_2 1$	$h_1h_2 5$	$h_2 4$	$h_1 1$
3	$h_1^{-1}h_3^{-1}5$	$h_2 1$	4	$h_1^{-1}2$	$h_2^{-1}h_1^{-1}h_3^{-1}h_2^{-1}h_1 6$	$h_2h_1^{-1}4$
4	$h_1 1$	$h_2^{-1}h_1^{-1}h_3^{-1}5$	$h_1 3$	$h_2^{-1}h_1^{-1}h_3^{-1}h_2^{-1}h_1 6$	2	$h_2^{-1}3$
5	$h_2^{-1}2$	$h_3h_1 4$	$h_2^{-1}h_1 6$	$h_3h_1^2 3$	$h_2^{-1}1$	6
6	$h_3h_1 4$	$h_2^{-1}h_1^{-1}2$	$h_3h_1h_2^{-1}h_1^{-1}h_3^{-1}5$	$h_1^{-1}1$	$h_3h_1^2h_2^{-1}3$	$h_1^{-1}5$

As in 2.3 the relations for H may then be shown to be $h_1h_2 = h_2h_1$, $h_2h_3h_2 = h_3$, $h_1h_3h_1 = h_3$. Thus $[h_1, h_2] = 1$, $[h_1, h_3^2] = 1$ and $[h_2, h_3^2] = 1$. H is normal in G and the factor group G/H is isomorphic to S_3. Further if $K = \langle h_1, h_2, h_3^2 \rangle$ then H/K is isomorphic to $\mathbb{Z}_2$ and K is $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

Alternatively we may show that G_6 is infinite by showing that the relations of G_6 hold in the infinite dihedral group D_∞ where $D_\infty \cong \langle a_1, a_2 \mid a_1^2 = a_2^2 = 1 \rangle$.

Let $a = a_1$ and $b = a_2$, then

$$c = ab = a_1a_2,$$
$$d = bc = a_2a_1a_2,$$
$$e = cd = a_1a_2a_2a_1a_2 = a_2,$$
$$f = de = a_2a_1a_2a_2 = a_2a_1.$$

It remains to show that the other two relations of G_6, namely $a = ef$ and $b = fa$, are satisfied.

Now,

$$ef = a_2a_2a_1 = a_1 = a,$$

and

$$fa = a_2a_1a_1 = a_2 = b.$$

Thus the relations of G_6 hold in D_∞ and so G_6 has D_∞ as a homomorphic image. Note that this result may be immediately generalised to show that G_n is infinite if $n \equiv 0 \bmod 6$.

In (21) R. C. Lyndon considered $A_n = G_n/G_n'$, proving that A_n is finite of order a_n where $a_n = f_n - 1 - (-1)^n$, $\{f_n\}$ being the Fibonacci type sequence 1,3,4,7,11,18 .... . The name of Fibonacci, in reality Leonardo of Pisa, is thus attached to a class of groups which with certain generalisations we consider in this chapter.

The Fibonacci groups $\underset{\sim}{F}(r, n)$ are defined in a paper by Johnson, Wamsley and Wright (31), the groups G_n being the Fibonacci groups $\underset{\sim}{F}(2, n)$. In 1971 I had, independently of Johnson, Wamsley and Wright, started to consider Fibonacci groups. In a search for a generalisation of the Conway problem I was able to show that, in the notation of (31), $\underset{\sim}{F}(3, 5)$ was $\mathbb{Z}_{22}$, the cyclic group of order 22. Although, as shown above, $\underset{\sim}{F}(2, 6)$ was infinite, it was an obvious question to ask whether $\underset{\sim}{F}(3, 6)$ was finite or infinite. Using the coset enumeration programme of M. J. Beetham (1) I was able to show that the cyclically presented group $\underset{\sim}{F}(3, 6)$ is a non-metacyclic group of order 1512. As far as I know this is the only finite non-metacyclic Fibonacci group known.

In Chapter 1 we explained what is meant by the cyclically presented group $G_n(w)$. In this chapter we look at some examples of the word w. When for some integer $r \geq 1$, the word w is given by

$$w = a_1 a_2 \dots a_r a_{r+1}^{-1}$$

then $G_n(w)$ is the *Fibonacci group* $\underset{\sim}{F}(r, n)$. We consider the following generalisation.

When for some integers $r,s,h \geq 1$, $k \geq 0$ the word w is given by

$$w = a_h\, a_{2h} \cdots a_{rh} \left(a_{rh+k}\, a_{(r+1)h+k} \cdots a_{(r+s-1)h+k}\right)^{-1}$$

we have a class of *groups of Fibonacci type* $\underset{\sim}{H}(r,n,k,s,h)$. (Note that we may assume without loss of generalisty that $r \geq s$). When $s=k=h=1$ we are back in the case of the Fibonacci groups, but when $s=h=1$ we have a class of groups referred to in the literature as the *generalised Fibonacci groups* (9),(10),(11),(13),(14),(29). This class we shall denote by $\underset{\sim}{F}(r,n,k)$. Thus

$$\underset{\sim}{F}(r,n,k) \cong \underset{\sim}{H}(r,n,k,1,1).$$

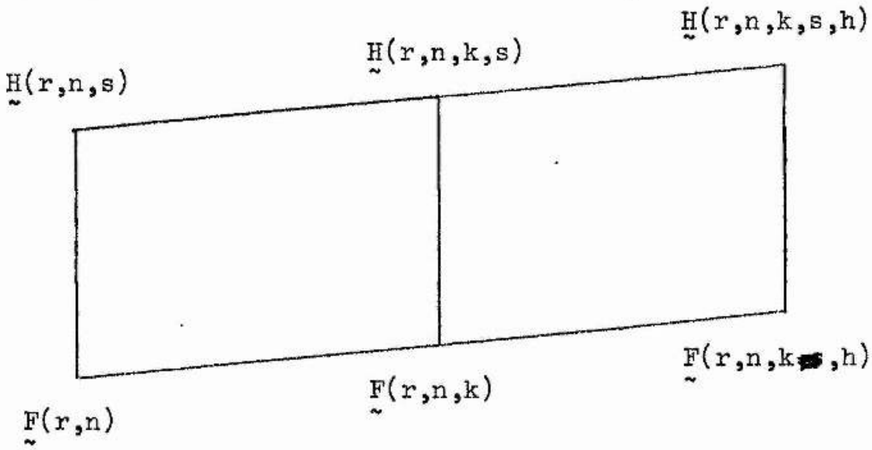
Special mention is also made of the groups

$$\underset{\sim}{F}(r,n,k,h) \cong \underset{\sim}{H}(r,n,k,1,h),$$

$$\underset{\sim}{H}(r,n,s) \cong \underset{\sim}{H}(r,n,1,s,1),$$

and $$\underset{\sim}{H}(r,n,k,s) \cong \underset{\sim}{H}(r,n,k,s,1).$$

The relationship between these classes of groups is shown in the following diagram.



The groups $\underset{\sim}{F}(2,n,n-1,2)$ are discussed in (25) while the groups $\underset{\sim}{F}(2,n,k,h)$ have been investigated by Johnson and Mawdesley (30).

As in Chapter 3 we consider first some isomorphisms between groups and some homomorphisms that occur. Then we give results where the orders of the groups are known. Next the structure of the two non-metacyclic groups $\underset{\sim}{F}(3, 6)$ and $\underset{\sim}{F}(3,6,5,2)$, both of order 1512, is examined in some detail, it being shown that neither is isomorphic to $\underset{\sim}{G}(2,3,-2)$ the non-metacyclic group of order 1512 discussed in 3.4.6. This is followed by a section on computing results where the orders of some groups are obtained by using the coset enumeration programme of M. J. Beetham (1).

We then consider the material of (10) and (11) and show that if $r \equiv 1 \bmod n$ and k and n are coprime then $\underset{\sim}{F}(r,n,k)$ is metacyclic of order r^n-1. Also $\underset{\sim}{F}(r,n,k)$ has a 2-generator 2-relation presentation

$$\langle x,y \mid y^{-1}xy = x^r, \quad y^n = x^{(r^n-1)/(n(r-1))} \rangle .$$

In (31) two questions are posed relating to the Fibonacci groups for the case $r \equiv 1 \bmod n$, namely to find 2-generator 2-relation presentations for them and also to find their orders. These two problems are thus solved, being special cases of the above results.

The results are then generalised and a proof is given that $\underset{\sim}{H}(r,n,k,s)$ is metacyclic if (i) $r \equiv s \bmod n$, (ii) $(r,n) = 1$, (iii) $(r+k-1,n) = 1$ and a 2-generator 2-relation presentation is found for these groups. Further if (iv) $(r,s) = 1$, then we show that $\underset{\sim}{H}(r,n,k,s)$ is a finite metacyclic group of order r^n-s^n. Note that when condition (iv) is added, (ii) becomes redundant. The result generalises the result given in (15) but, since the proof of (15) does not generalise, the proof given is different.

In a similar way to that in which the results of (10), (11) are extended in (14) to the class $\underset{\sim}{F}(r,n,k,h)$, the results of the above paragraph are generalised to the groups $\underset{\sim}{H}(r,n,k,s,h)$. Finally E. F. Robertson and I conjecture in (15) that the groups $\underset{\sim}{H}(r,4,2)$, r odd are metacyclic. In this chapter it is shown how these groups have 2-generator 2-relation presentations. The conjecture has since been proved by A. M. Brunner (6).

4.2. Some isomorphisms and homorphisms, results on orders.

In this section we begin by considering some elementary isomorphisms, the proofs of which are straightforward. We then give some homorphisms and also consider when $\underset{\sim}{H}(r,n,k,s,h)/\underset{\sim}{H}'(r,n,k,s,h)$, the factor by the derived group, is infinite. The section concludes by showing that when $r \equiv 2,3 \bmod 4$ then $\underset{\sim}{H}(r,4,r-1)$ is isomorphic to $\mathbb{Z}_5$ and when $r \equiv 3,4 \bmod 6$ then $\underset{\sim}{H}(r,6,r-1)$ is isomorphic to $\mathbb{Z}_{13}$.

It follows immediately from the definition that if $k_1 \equiv k_2 \bmod n$ and $h_1 \equiv h_2 \bmod n$ then

$$\underset{\sim}{H}(r,n,k_1,s,h_1) \cong \underset{\sim}{H}(r,n,k_2,s,h_2)$$

so that when we write $\underset{\sim}{H}(r,n,k,s,h)$ we shall assume that k and h have been reduced mod n. We may also assume without loss of generality that $r \geq s$. If $r = s$, then $\underset{\sim}{H}(r,n,k,s,h)$ has the infinite cyclic group as a homomorphic image and so is infinite. This is immediate on taking $a_1 = a_2 = \ldots = a_n$. In what follows we shall therefore assume that $r > s$.

<u>Lemma 4.2.1.</u> $\underset{\sim}{H}(r,n,k,s,h) \cong \underset{\sim}{H}(r,n,(r+s-2)h + k,s, - h)$,

$$\cong \underset{\sim}{H}(r,n, - k,s, - h),$$

$$\cong \underset{\sim}{H}(r,n, - (r+s-2)h - k,s,h).$$

<u>Proof</u>. Consider the maps ϕ_1, ϕ_2, ϕ_3 from the free group F_n on $\{x_i : i \in \mathbb{Z}_n\}$ to $\underset{\sim}{H}(r,n,k,s,h)$ induced by $x_i\phi_1 = a_i^{-1}, x_i\phi_2 = a_{-i}$ and $x_i\phi_3 = a_{-i}^{-1}$. Since

$$(x_h x_{2h} \ldots x_{rh}(x_{rh+k}\, x_{(r+1)h+k} \ldots x_{(r+s-1)h+k})^{-1})\phi_1 =$$

$$a_h^{-1}\, a_{2h}^{-1} \ldots a_{rh}^{-1}(a_{rh+k}^{-1}\, a_{(r+1)h+k}^{-1} \ldots a_{(r+s-1)h+k}^{-1})^{-1},$$

then $\underset{\sim}{H}(r,n,k,s,h)$ is isomorphic to the group with presentation

$$\langle a_1, a_2, \ldots, a_n \mid (a_{rh}\, a_{(r-1)h} \ldots a_{2h} a_h (a_{(r+s-1)h+k} \ldots a_{(r+1)h+k} a_{(rh+k)})^{-1})\theta^{i-1} = 1,$$

$$1 \le i \le n \rangle,$$

where θ is the automorphism of F_n induced by the permutation $(1\ 2 \ldots n)$ of $\mathbb{Z}_n$. Thus

$$\underset{\sim}{H}(r,n,k,s,h) \cong \underset{\sim}{H}(r,n,(r+s-2)h+k,s,-h).$$

Similarly,

$$(x_h\, x_{2h} \ldots x_{rh}(x_{rh+k} x_{(r+1)h+k} \ldots x_{(r+s-1)h+k})^{-1})\phi_2 =$$

$$a_{-h}\, a_{-2h} \ldots a_{-rh}(a_{-rh-k} a_{-(r+1)h-k} \ldots a_{-(r+s-1)h-k})^{-1},$$

and

$$(x_h\, x_{2h} \ldots x_{rh}(x_{rh+k} x_{(r+1)h+k} \ldots x_{(r+s-1)h+k})^{-1})\, \phi_3 =$$

$$a_{-h}^{-1}\, a_{-2h}^{-1} \ldots a_{-rh}^{-1}\, (a_{-rh-k}^{-1}\, a_{-(r+1)h-k}^{-1} \ldots a_{-(r+s-1)h-k}^{-1})^{-1},$$

give

$$\underset{\sim}{H}(r,n,k,s,h) \cong \underset{\sim}{H}(r,n,-k,s,-h),$$

and

$$\underset{\sim}{H}(r,n,k,s,h) \cong \underset{\sim}{H}(r,n,-(r+s-2)h-k,s,h),$$

respectively. Alternatively the third isomorphism may be obtained by combining the first two isomorphisms.

Lemma 4.2.2. If α is coprime to n,

$$\underset{\sim}{H}(r,n,k,s,h) \cong \underset{\sim}{H}(r,n,k/\alpha,s,h/\alpha).$$

Proof. This isomorphism is immediate on considering the map ϕ from the free group on $\{x_i : i \in \mathbb{Z}_n\}$ to $\underset{\sim}{H}(r,n,k,s,h)$ induced by $x_i\phi = a_{i/\alpha}$.

Notice that it follows from this result that if h is coprime to n,

$$\underset{\sim}{H}(r,n,k,s,h) \cong \underset{\sim}{H}(r,n,k/h,s)$$

and, in the special case of the groups $\underset{\sim}{F}(r,n,k,h)$,

$$\underset{\sim}{F}(r,n,k,h) \cong \underset{\sim}{F}(r,n,k/h).$$

Lemma 4.2.3. $\underset{\sim}{H}(1,n,k,1,h) \cong F_d$, the free group of rank d, where $d = (n,k)$.

Proof. $\underset{\sim}{H}(1,n,k,1,h) \cong \langle a_1,a_2, \ldots, a_n \mid (a_h a_{h+k}^{-1})\theta^{i-1}=1,\ 1 \le i \le n \rangle$ from which we get that $\underset{\sim}{H}(1,n,k,1,h)$ is the free group generated by $a_1,a_2, \ldots, a_d$.

Theorem 4.2.4. If $(n,k,h) = d \neq 1$, then

$$\underset{\sim}{H}(r,n,k,s,h) \cong \underset{d}{*}\ \underset{\sim}{H}(r,n/d,k/d,s,h/d),$$

the free product of d copies of $\underset{\sim}{H}(r,n/d,k/d,s,h/d)$.

Proof. Let $\alpha = n/d$, $\beta = k/d$, $\gamma = h/d$ and fix t with $0 \le t \le d-1$. With $x_j = a_{jd+t}$ the relations (id+t), $1 \le i \le \alpha$, reduce to

$$((x_\gamma\, x_{2\gamma}\cdots x_{r\gamma})(x_{r\gamma+\beta}\, x_{(r+1)\gamma+\beta}\cdots x_{(r+s-1)\gamma+\beta})^{-1})\bar{\theta}^{\,i-1} = 1, \quad 1 \le i \le \alpha,$$

where the subscripts of the x_i are reduced mod α and permuted by $\bar{\theta}$ according to the permutation $(1\ 2\ \dots\ \alpha)$. The result is now immediate.

<u>Lemma 4.2.5</u> $\underset{\sim}{H}(r,1,k,s,h) \cong \mathbb{Z}_{r-s}$.

<u>Proof</u> $\underset{\sim}{H}(r,1,k,s,h) \cong \langle\, a \,|\, a^r = a^s \,\rangle$,

and so $\underset{\sim}{H}(r,1,k,s,h) \cong \mathbb{Z}_{r-s}$ if $r > s$,

and is infinite if $r = s$.

<u>Theorem 4.2.6</u>. If $rh + k \equiv h \bmod n$ or $sh + k \equiv h \bmod n$, then $\underset{\sim}{H}(r,n,k,s,h)$ is infinite if $(r-s) \neq 1$ and $(r-s)h$ is not coprime to n and $\underset{\sim}{H}(r,n,k,s,h)$ is isomorphic to $\mathbb{Z}_{r-s}$ if $(r-s)h$ is coprime to n or $r-s=1$.

<u>Proof</u>. $\underset{\sim}{H}(r,n,k,s,h) \cong \langle a_1, a_2, \dots, a_n \,|\, (a_h a_{2h} \cdots a_{rh}(a_{rh+k} a_{(r+1)h+k} \cdots$

$$a_{(r+s-1)h+k})^{-1})\theta^{i-1} = 1,\ 1 \le i \le n \,\rangle.$$

If $rh + k \equiv h \bmod n$ or $sh + k \equiv h \bmod n$, then cancellation occurs and the presentation for $\underset{\sim}{H}(r,n,k,s,h)$ may be written as

$$\langle a_1, a_2, \dots, a_n \,|\, (a_h\, a_{2h} \cdots a_{(r-s)h})\theta^{i-1} = 1,\ 1 \le i \le n \,\rangle.$$

Assume that $r-s \neq 1$.
From the relations

$$a_n\, a_h\, a_{2h} \cdots a_{(r-s-1)h} = 1,$$

and

$$a_h\, a_{2h} \cdots a_{(r-s)h} = 1,$$

we obtain

$$a_{(r-s)h} = a_n,$$

and, in general,

$$a_{ih} = a_{(r-s+i)h}, \quad 1 \le i \le n. \qquad (*)$$

Also

$$a_{ih+\alpha} = a_{(r-s+i)h+\alpha}, \quad 1 \le i \le n,\ 0 \le \alpha \le h-1.$$

Let $d = ((r-s)h,n)$. Then the relations (*) give immediately

$$a_{(r-s)h} = a_{2(r-s)h} = \ldots = a_{(n/d)(r-s)h}.$$

Therefore the n generators a_i are partitioned into d classes, the n/d members of each class being equal. The n relations now reduce to

$$(a_h \, a_{2h} \, \ldots \, a_{\frac{dh}{(d,h)}})^{(r-s)(d,h)/d} = 1.$$

When $d \neq 1$, ~~$a_{dh} \neq a_h$ so we may add the relation~~ then either d divides h or $d/(d,h) \neq 1$. In both cases the group is infinite.

$$\cancel{a_{dh} = (a_h \, a_{2h} \, \ldots \, a_{(d-1)h})^{-1}}$$

~~to obtain F_{d-1}, the free group of rank d-1, as a homomorphic image of $\underset{\sim}{H}(r,n,k,s,h)$ and thus $\underset{\sim}{H}(r,n,k,s,h)$ is infinite~~.

If, however, $d = 1$ then,

$$\underset{\sim}{H}(r,n,k,s,h) \cong \langle a_h \mid a_h^{r-s} = 1 \rangle,$$

and so $\underset{\sim}{H}(r,n,k,s,h)$ is isomorphic to $\mathbb{Z}_{r-s}$.

If $r-s=1$ the group is clearly trivial.

<u>Lemma 4.2.7</u>. $\underset{\sim}{H}(r,n,k,s,h)$ has $\underset{\sim}{H}(r,m,k,s,h)$ as a homomorphic image if m divides n.

<u>Proof</u>. The result follows by adding the relations

$a_i \, a_{i+m}^{-1} = 1$, $i \in \mathbb{Z}_n$ to the relations for $\underset{\sim}{H}(r,n,k,s,h)$.

<u>Theorem 4.2.8</u>. $\underset{\sim}{H}(r,n,k,s,h)$ is infinite if $(r,n,s,h-k) \neq 1$.

<u>Proof</u>. Let $d = (r,n,s,h-k) \neq 1$. By Lemma 4.2.7 $\underset{\sim}{H}(r,n,k,s,h)$ has $\underset{\sim}{H}(r,d,k,s,h)$ as a homomorphic image. But $\underset{\sim}{H}(r,d,k,s,h)$ is infinite by Theorem 4.2.6 since d divides r,s and $(h-k)$. Therefore $\underset{\sim}{H}(r,n,k,s,h)$ is infinite.

Theorem 4.2.9. $H(r, n, s)$ is infinite if

(i) $r + s \equiv 0 \mod n, \quad n \geq 5,$

or (ii) $n = 4$ and $r + s$ is divisible by 8.

Proof. If $H(r, n, s)$ is finite it is a group of deficiency zero and so, by Theorem 1.3.3, it follows that $H(r, n, s)$ is infinite if the commutator quotient has rank ≥ 4. Consider A the factor group of the commutator quotient of $H(r, n, s)$ obtained by adding the relations $a_i^2 = 1$, $1 \leq i \leq n$. Since $r + s \equiv 0 \mod n$ the relations for $H(r, n, s)$ become in A the single relation

$$a_1a_2 \ldots a_r = a_{r+1}a_{r+2} \ldots a_n,$$

that is

$$a_1a_2 \ldots a_{n-1} = a_n.$$

Therefore A is isomorphic to the direct sum of n-1 copies of $\mathbb{Z}_2$, and so, if $n \geq 5$ the commutator quotient has rank ≥ 4.

If, however, $n = 4$ and $r + s$ is divisible by 8, then the relations for $H(r, 4, s)$ become redundant in A and so A has rank 4. Thus in this case also, $H(r, n, s)$ is infinite.

Theorem 4.2.10. For n odd and h such that $(3h, n) = 1$, $F(2, n, -h/2, h)$ is perfect, having $SL(2, p)$ as a homomorphic image if p is a prime divisor of n, $p \geq 5$.

Proof. The relations of $F(2, n, -h/2, h)$ are $a_1a_{1+h} = a_{1+h/2}$, $a_2a_{2+h} = a_{2+h/2}$, $\ldots$, $a_{n-1}a_{n-1+h} = a_{n-1+h/2}$, $a_na_h = a_{h/2}$.

Now $a_{h/2}a_{3h/2} = a_h$ and therefore

$$(a_na_h)(a_ha_{2h}) = a_{h/2}a_{3h/2} = a_h.$$

Similarly,

$$(a_ha_{2h})(a_{2h}a_{3h}) = a_{2h}.$$

If $A(2, n, -h/2, h)$ denotes the commutator quotient of $\underset{\sim}{F}(2, n, -h/2, h)$ then in $A(2, n, -h/2, h)$ we have

$$a_n a_h a_{2h} = 1,$$

and

$$a_h a_{2h} a_{3h} = 1.$$

Thus $a_n = a_{3h}$. Since n is coprime to 3h, $A(2, n, -h/2, h)$ is the trivial group and so $\underset{\sim}{F}(2, n, -h/2, h)$ is perfect.

Now take $x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$ where the entries in the matrices are in $GF(p)$, and put $a_i = x^{-i}y^{-1/h}x^{i+h/2}$. Then x and y generate $SL(2, p)$. It is straightforward to show that

$$x^{-i}y^{-1/h}x^{h/2+i}x^{-i-h}y^{-1/h}x^{h/2+h+i} = x^{-(i+h/2)}y^{-1/h}x^{i+h}, \qquad 1 \leq i \leq p,$$

that is

$$x^{h/2}y^{-1/h}x^{-h/2}y^{-1/h}x^{h/2}y^{1/h} = 1.$$

Therefore $a_i a_{i+h} = a_{i+h/2}$, $1 \leq i \leq p$ and thus $a_1, a_2, \ldots, a_p$ satisfy the relations for $\underset{\sim}{F}(2, p, -h/2, h)$.

A consequence of this theorem is the following result. Since coset enumeration gives $|\underset{\sim}{F}(2, 5, 2)| = 120$, then $\underset{\sim}{F}(2, 5, 2)$ is isomorphic to $SL(2, 5)$. However M. J. Dunwoody (25) has shown that if $p > 5$, $\underset{\sim}{F}(2, p, -\tfrac{1}{2})$ is infinite.

Next we consider $\underset{\sim}{H}(r, n, k, s, h)/\underset{\sim}{H}'(r, n, k, s, h)$. D. L. Johnson has shown in (28) that $\underset{\sim}{F}(r, n)/\underset{\sim}{F}'(r, n)$ is finite and this is extended in (31) where it is shown that under certain conditions $\underset{\sim}{F}(r, n, k)/\underset{\sim}{F}'(r, n, k)$ is also finite. We extend these results to the class $\underset{\sim}{H}(r, n, k, s, h)$. We shall require the idea of the polynomial associated with the word w. Associate with $G_n(w)$, see 1.3, a polynomial $f(x)$ where

$$f(x) \equiv b_0 + b_1 x + b_2 x^2 + \ldots + b_{n-1}x^{n-1}$$

where b_{i-1} is the sum of the exponents of a_i in w. Then $f(x) \in \mathbb{Z}[x]$ and call $f(x)$ the polynomial associated with w. Before we give the theorem we state and prove the following lemma.

<u>Lemma 4.2.11.</u> There exists an *nth* root of unity ξ with $\xi^n = 1$, $\xi^\ell = 1$, $\xi^m = -1$, $\xi^p \neq 1$ if, and only if,

(i) $\nu_2(n) > \nu_2(m)$ and $\nu_2(\ell) > \nu_2(m)$,

and (ii) *either* $\nu_2(m) \geq \nu_2(p)$ *or* (n,ℓ,m) does not divide $2^{\nu_2(n,\ell,m)}p$,

where ν_2 denotes the 2-part of a positive integer and $\nu_2(0) = \infty$.

<u>Proof.</u> Let $\ell' = \ell/2^{\nu_2(\ell)}$ ~~$\ell/\nu_2(\ell)$~~, $m' = m/2^{\nu_2(m)}$ ~~$m/\nu_2(m)$~~, $n' = n/2^{\nu_2(n)}$ ~~$n/\nu_2(n)$~~ and suppose that $\xi^n = 1$, $\xi^\ell = 1$, $\xi^m = -1$, $\xi^p \neq 1$. Since $\xi^n = 1$ we may take $\xi = e^{2t\pi i/n}$. Then, since $\xi^m = -1$, $2tm/n$ is odd and so $\nu_2(n) > \nu_2(m)$ and $t = \left(2^{\nu_2(n)-\nu_2(m)-1}\right.$. $\left.\frac{n'(2r+1)}{(n',m')}\right)$ for some $r \in \mathbb{Z}$. However we also have $\xi^\ell = 1$ and so $t\ell/n \in \mathbb{Z}$. Therefore $2^{\nu_2(n)-\nu_2(m)-1}\,\dfrac{n'(2r+1)\ell}{(n',m')n} = \dfrac{(2r+1)\ell}{(n',m')2^{\nu_2(m)+1}} \in \mathbb{Z}$. Hence

$\nu_2(\ell) > \nu_2(m)$ and r is such that $(2r+1)\ell/(n',m') \in \mathbb{Z}$. Since $\xi^p \neq 1$, $tp/n \notin \mathbb{Z}$, that is $2^{\nu_2(n)-\nu_2(m)-1}\,\dfrac{n'(2r+1)p}{(n',m')n} = \dfrac{(2r+1)p}{(n',m')2^{\nu_2(m)+1}} \notin \mathbb{Z}$.

Therefore either $\nu_2(m) \geq \nu_2(p)$ or (n',m') does not divide $(2r+1)p$. But we can choose $r \in \mathbb{Z}$ such that $2r+1 = (n',m')/(n',m',\ell')$. Hence (n',m') does not divide $(n',m')p/(n',m',\ell')$, that is (n',m',ℓ') does not divide p or, alternatively, (n,ℓ,m) does not divide $2^{\nu_2(n,\ell,m)}p$.

Conversely, assume that ~~either~~ both condition (i) ~~or~~ and condition (ii) hold~~s~~ and let $\xi = e^{\frac{\pi i}{(n',\ell',m')2^{\nu_2(m)}}}$. Then $\xi^n = 1$, $\xi^\ell = 1$, $\xi^m = -1$ and $\xi^p \neq 1$.

<u>Theorem 4.2.12.</u> $\underset{\sim}{H}(r, n, k, s, h)/\underset{\sim}{H}'(r, n, k, s, h)$ is infinite if, and only if, at least one of the following three conditions holds:

(i) (rh, sh, n) does not divide h,

(ii) $((r-1)h+k, (s-1)h+k, n)$ does not divide h,

(iii) $\nu_2((r+s)h)$ and $\nu_2(n) > \nu_2(k-h)$ and further *either* $\nu_2(k-h) \geq \nu_2(h)$ *or* $(n, k-h, (r+s)h)$ does not divide $2^{\nu_2((n,k-h,(r+s)h)}h$, where ν_2 denotes the 2-part of a positive integer and $\nu_2(0) = \infty$.

<u>Proof.</u> The polynomial associated with H(r, n, k, s, h)/H'(r, n, k, s, h) is f(x) where

$$f(x) \equiv 1 + x^h + x^{2h} + \ldots + x^{(r-1)h} - x^{(r-1)h+k} - x^{rh+k} - \ldots - x^{(r+s)h+k-2h}$$

and by Theorem 2 of (29), $\underset{\sim}{H}(r, n, k, s, h)$ has an infinite abelian factor precisely when there is an nth root of unity ξ satisfying $f(\xi) = 0$. Now $f(\xi) \neq 0$ when $\xi^h = 1$. For suppose $f(\xi) = 0$ and $\xi^h = 1$. Then

$$1 + \xi^h + \xi^{2h} + \ldots + \xi^{(r-1)h} - \xi^{(r-1)h+k} - \xi^{rh+k} - \ldots - \xi^{(r+s)h+k-2h} = 0,$$

that is

$$r - s\xi^k = 0,$$

which implies that $\xi^k = r/s$. But $|\xi^k| = 1$ so $r = s$ since r and s are both positive, which is a contradiction.

Hence $\xi^h \neq 1$.

$$(1 - \xi^h)f(\xi) = (1 + \xi^h + \xi^{2h} + \ldots + \xi^{(r-1)h} - \xi^{(r-1)h+k} - \xi^{rh+k} - \ldots - \xi^{(r+s)h+k-2h})$$

$$- (\xi^h + \xi^{2h} + \ldots + \xi^{rh} - \xi^{rh+k} - \ldots - \xi^{(r+s)h+k-h}),$$

$$= 1 - \xi^{rh} - \xi^{(r-1)h+k} + \xi^{(r+s)h+k-h}.$$

Therefore $(1 - \xi^h)f(\xi) = 0$ when

$$1 - \xi^{rh} = \xi^{(r-1)h+k}(1 - \xi^{sh}). \tag{1}$$

Hence $|1 - \xi^{rh}| = |\xi^{(r-1)h+k}||1 - \xi^{sh}| = |1 - \xi^{sh}|$, $\xi^h \neq 1$ and so $\xi^{rh} = \xi^{sh}$ or $\xi^{rh} = \xi^{-sh}$.

Substituting $\xi^{rh} = \xi^{sh}$ in (1) gives

$$1 - \xi^{rh} = \xi^{(r-1)h+k}(1 - \xi^{rh}),$$

so

$$(1 - \xi^{rh})(1 - \xi^{(r-1)h+k}) = 0. \tag{2}$$

Hence $\xi^{rh} = \xi^{sh} = 1$, $\xi^{n} = 1$, $\xi^{h} \neq 1$ giving (rh, sh, n) does not divide h which is condition (i) or $\xi^{(r-1)h+k} = \xi^{(s-1)h+k} = 1$, $\xi^{n} = 1$, $\xi^{h} \neq 1$ giving $((r-1)h+k, (s-1)h+k, n)$ does not divide h which is condition (ii).

The other case is when $\xi^{rh} = \xi^{-sh}$. Then substituting $\xi^{rh} = \xi^{-sh}$ in (1) gives

$$1 - \xi^{rh} = \xi^{(r-1)h+k} - \xi^{k-h} ,$$

that is

$$(1 - \xi^{rh})(1 + \xi^{k-h}) = 0,$$

Either $\xi^{rh} = 1$ and so $\xi^{rh} = \xi^{sh} = 1$, $\xi^{n} = 1$, $\xi^{h} \neq 1$ which again gives condition (i) or $\xi^{rh} = \xi^{-sh}$, $\xi^{k-h} = -1$, $\xi^{n} = 1$, $\xi^{h} \neq 1$ and so by Lemma 4.2.11 we require that $\nu_2((r+s)h)$ and $\nu_2(n)$ are each greater than $\nu_2(k-h)$ and further that either $\nu_2(k-h) \geq \nu_2(h)$ or $(n, k-h, (r+s)h)$ does not divide $2^{\nu_2((n,k-h,(r+s)h))}h$. This gives condition (iii).

For the converse, let $d \neq 1$ be ~~a~~ the highest common factor of rh, sh and n. Since d does not divide h, then if ξ is a primitive dth root of unity $\xi^{h} \neq 1$, $\xi^{n} = 1$, $\xi^{rh} = 1$, $\xi^{sh} = 1$ and equation (1) is satisfied. If condition (ii) holds let $((r-1)h+k, (s-1)h+k, n) = d'$. Since d' does not divide h, then if ξ is a primitive $d'th$ root of unity, $\xi^{h} \neq 1$, $\xi^{n} = 1$, $\xi^{(r-1)h+k} = \xi^{(s-1)h+k} = 1$ and again equation (1) is satisfied. Finally if condition (iii) holds, let $d = \left\{n\, 2^{\nu_2(k-h)-\nu_2(n)+1}, 2(k-h), (r+s)h\, 2^{\nu_2(k-h)-\nu_2((r+s)h)+1}\right\}$ and let ξ be a primitive dth root of unity. Then by Lemma 4.2.11 $\xi^{(r+s)h} = 1$, $\xi^{(k-h)} = -1$, $\xi^{n} = 1$, $\xi^{h} \neq 1$ and again equation (1) is satisfied.

The two corollaries which follow are the two special cases of this theorem discussed in Theorem 3 of (29) and Theorem 1 of (15) respectively.

Corollary 1. $\underset{\sim}{F}(r, n, k)/\underset{\sim}{F}'(r, n, k)$ is infinite if, and only if, at least one of the following two conditions holds:

(i) $(r-1, k, n) \neq 1$,

(ii) $\nu_2(r+1)$ and $\nu_2(n)$ are each greater than $\nu_2(k-1)$, where ν_2 denotes the 2-part of a positive integer and $\nu_2(0) = \infty$.

Proof. Let $s = h = 1$ in Theorem 4.2.12. Then condition (i) never occurs and conditions (ii) and (iii) reduce to the statement of the corollary.

Corollary 2. $\underset{\sim}{H}(r, n, s)/\underset{\sim}{H}'(r, n, s)$ is infinite if, and only if, $(r, n, s) \neq 1$.

Proof. Let $k = h = 1$ in Theorem 4.2.12 and the result is immediate.

It is the main aim of the remaining lemmas and theorems in this section to show that if $r \equiv 2, 3 \bmod 4$ then $\underset{\sim}{H}(r, 4, r-1) \cong \mathbb{Z}_5$ and if $r \equiv 3, 4 \bmod 6$ then $\underset{\sim}{H}(r, 6, r-1) \cong \mathbb{Z}_{13}$. In the course of proving this result we state and prove some results that are more general than is strictly necessary for its proof. We begin by considering $\underset{\sim}{H}(r, 2n, r-1)/\underset{\sim}{H}'(r, 2n, r-1)$.

Theorem 4.2.13. If $r \equiv n \bmod 2n$, $|\underset{\sim}{H}(r, 2n, r-1)/\underset{\sim}{H}'(r, 2n, r-1)| = A_n$, where $A_1 = 1$ and $A_n = 3A_{n-1} + 2(-1)^n$, $n > 1$.

Proof. $\underset{\sim}{H}(r, 2n, r-1) \cong \langle a_1, a_2, \ldots, a_{2n} \mid (a_1 a_2 \ldots a_r)(a_{r+1}\, a_{r+2} \ldots a_{2r-1})^{-1})\theta^{i-1} = 1,\ 1 \leq i \leq 2n \rangle$.

From relations (1) and (1+n)

$$a_1a_2 \cdots a_r = a_{r+1}a_{r+2} \cdots a_{2r-1},$$

and

$$a_{n+1}a_{n+2} \cdots a_{n+r} = a_{n+r+1}a_{n+r+2} \cdots a_{n+2r-1}.$$

But since $r \equiv n \bmod 2n$, $n + r \equiv 0 \bmod 2n$ and so

$$a_r a_{r+n} = 1, \qquad (\alpha)$$

Similarly

$$a_i a_{i+n} = 1, \quad 1 \leq i \leq n. \qquad (\beta)$$

If we abelianise the relations and make use of the equations (β) then we obtain the n relations

$$a_1^{\,2}a_2^{\,2} \cdots a_{n-1}^{\,2}\ a_n = 1,$$
$$a_1^{-1}a_2^{\,2}a_3^{\,2} \cdots \ a_n^{\,2} = 1,$$
$$a_1^{-1}a_2^{-1}a_3^{\,2} \cdots \ a_n^{\,2} = 1,$$
$$\cdots$$
$$a_1^{-2}a_2^{-2} \cdots a_{n-1}^{-1}a_n^{\,2} = 1.$$

If we number these equations (R1), (R2), ..., (Rn), then from (R1) and (R2)

$$a_n = a_1^{\,3},$$

from (R2) and (R3),

$$a_1 = a_2^{-3},$$

and, in general, from (Ri) and (R(i+1))

$$a_{i-1} = a_i^{-3}, \qquad 2 \leq i \leq n-1.$$

From (Rn) and (R1)

$$a_{n-1} = a_n^{-3}.$$

We therefore have the relations

$$a_n = a_1^{\ 3}$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad(\gamma)$

$$a_i = a_{i+1}^{-3}, \qquad 1 \leq i \leq n-1 .$$

From (R2)

$$\begin{aligned} a_1 &= (a_2^{\ 2} a_3^{\ 2} \ldots a_{n-1}^{\ 2} a_n) a_n, \\ &= (a_2^{\ 2} a_3^{\ 2} \ldots a_{n-1}^{\ 2} a_n) a_1^{\ 3}, \\ &= a_n^{(1 + 2(-3) + 2(-3)^2 + \ldots + 2(-3)^{n-2})} a_1^{\ 3}, \qquad (\delta) \end{aligned}$$

and $1 + 2 \sum_{i=1}^{n-2} (-3)^i$ is positive or negative according as n is ~~positive~~ even or ~~negative~~ odd. Also $(1 + 2(-3) + 2(-3)^2 + \ldots + 2(-3)^{n-2})$ is equal to A_{n-1} if n is even and to $-A_{n-1}$ if n is odd. Since $a_n = a_1^{\ 3}$, we have from (δ)

$$a_1^{\ 2} a_n^{(-1)^n A_{n-1}} = 1,$$

that is

$$A_n = 3 A_{n-1} + 2(-1)^n.$$

Lemma 4.2.14. If $r \equiv n \bmod 2n$ where $n = 2,3$ then $\underset{\sim}{H}'(r, 2n, r-1)$ is the trivial group.

Proof. Consider first the case where $r \equiv 2 \bmod 4$ and let $r = 4u + 2$. Then $\underset{\sim}{H}(r, 4, r-1)$ has a presentation on the four generators a_1, a_2, a_3, a_4 subject to the four relations

$$(a_1 a_2 a_3 a_4)^u a_1 a_2 = a_3 (a_4 a_1 a_2 a_3)^u, \qquad (1)$$

$$(a_2 a_3 a_4 a_1)^u a_2 a_3 = a_4 (a_1 a_2 a_3 a_4)^u, \qquad (2)$$

$$(a_3 a_4 a_1 a_2)^u a_3 a_4 = a_1 (a_2 a_3 a_4 a_1)^u, \qquad (3)$$

$$(a_4 a_1 a_2 a_3)^u a_4 a_1 = a_2 (a_3 a_4 a_1 a_2)^u. \qquad (4)$$

As in Theorem 4.2.13 we get from (1) and (3)

$$a_2 = a_4^{-1},$$

and from (2) and (4)

$$a_1 = a_3^{-1}.$$

From (3)

$$a_4(a_3a_4a_1a_2)^u a_3a_4 = a_4a_1(a_2a_3a_4a_1)^u, \tag{5}$$

from (2)

$$a_4a_3(a_2a_3a_4a_1)^u a_2a_3 = a_4a_3a_4(a_1a_2a_3a_4)^u, \tag{6}$$

from (1)

$$a_4a_3a_2(a_1a_2a_3a_4)^u a_1a_2 = a_4a_3a_2a_3(a_4a_1a_2a_3)^u, \tag{7}$$

and from (4)

$$a_4a_3a_2a_1(a_4a_1a_2a_3)^u a_4a_1 = a_4a_3a_2a_1a_2(a_3a_4a_1a_2)^u. \tag{8}$$

From (5), (6), (7) and (8)

$$a_4a_3a_2a_1(a_4a_1a_2a_3)^u a_4a_1 = a_4a_1(a_2a_3a_4a_1)^u,$$

that is

$$a_4a_3a_2a_1 = 1. \tag{9}$$

But $a_4 = a_2^{-1}$ and $a_3 = a_1^{-1}$ and so (9) becomes

$$a_1a_2 = a_2a_1.$$

Therefore $\underset{\sim}{H}'(r, 4, r-1)$, where $r \equiv 2 \bmod 4$, is the trivial group.

In a similar way we can now consider the case $r \equiv 3 \bmod 6$ and let $r = 6v + 3$. Then $\underset{\sim}{H}(r, 6, r-1)$ has a presentation on the six generators $a_1, a_2, a_3, a_4, a_5, a_6$ subject to the six relations

$$(a_1a_2a_3a_4a_5a_6)^v a_1a_2a_3 = a_4a_5(a_6a_1a_2a_3a_4a_5)^v, \tag{10}$$

$$(a_2a_3a_4a_5a_6a_1)^v a_2a_3a_4 = a_5a_6(a_1a_2a_3a_4a_5a_6)^v, \tag{11}$$

$$(a_3a_4a_5a_6a_1a_2)^v a_3a_4a_5 = a_6a_1(a_2a_3a_4a_5a_6a_1)^v, \tag{12}$$

$$(a_4a_5a_6a_1a_2a_3)^v a_4a_5a_6 = a_1a_2(a_3a_4a_5a_6a_1a_2)^v, \tag{13}$$

$$(a_5a_6a_1a_2a_3a_4)^v a_5a_6a_1 = a_2a_3(a_4a_5a_6a_1a_2a_3)^v, \tag{14}$$

$$(a_6a_1a_2a_3a_4a_5)^v a_6a_1a_2 = a_3a_4(a_5a_6a_1a_2a_3a_4)^v. \tag{15}$$

As in Theorem 4.2.13

$$a_4 = a_1^{-1},\ a_5 = a_2^{-1} \quad \text{and} \quad a_6 = a_3^{-1}. \tag{16}$$

From relation (10),

$$a_1(a_2a_3a_4a_5a_6a_1)^v a_2a_3 = a_4a_5(a_6a_1a_2a_3a_4a_5)^v, \tag{17}$$

from relations (14) and (17)

$$a_1(a_5a_6a_1a_2a_3a_4)^v a_5a_6a_1 = a_4a_5(a_6a_1a_2a_3a_4a_5)^v, \tag{18}$$

from relation (12)

$$a_1a_5(a_3a_4a_5a_6a_1a_2)^v a_3a_4a_5 = a_4a_5(a_6a_1a_2a_3a_4a_5)^v, \tag{19}$$

that is

$$a_1a_5a_3 = 1. \tag{20}$$

Similarly from (11), (13), (15)

$$a_2a_6a_4 = 1. \tag{21}$$

From (16), (20), (21)

$$a_1a_2^{-1}a_3 = 1,\ a_2a_3^{-1}a_1^{-1} = 1,$$

that is

$$a_3 = a_2a_1^{-1},$$

and

$$a_1a_2 = a_2a_1.$$

Therefore $\underset{\sim}{H}(r, 6, r-1)$, where $r \equiv 3 \bmod 6$, is abelian.

Lemma 4.2.15. If $r \equiv n \bmod 2n$, then $|\underset{\sim}{H}(r+1,\ 2n,\ r)/\underset{\sim}{H}'(r+1,\ 2n,\ r)| = |\underset{\sim}{H}(r,\ 2n,\ r-1)/\underset{\sim}{H}'(r,\ 2n,\ r-1)|$.

Proof. $\underset{\sim}{H}(r+1,\ 2n,\ r) \cong \langle a_1,\ a_2,\ \ldots,\ a_{2n} | ((a_1 a_2 \ldots a_{r+1})(a_{r+2} a_{r+3} \ldots a_{2r+1})^{-1})\theta^{i-1} = 1,\ 1 \leq i \leq 2n \rangle$. The proof is similar to that of Lemma 4.2.13. It is easy to show that

$$a_i a_{i+n} = 1, \qquad 1 \leq i \leq n. \qquad (*)$$

If we abelianise the relations for $\underset{\sim}{H}(r+i,\ 2n,\ r)$ and make use of the equations (*), then we obtain the n relations

$$a_1^{-1} a_2^{2} \ldots a_n^{2} = 1,$$
$$a_1^{-2} a_2^{-1} \ldots a_n^{2} = 1,$$
$$\ldots$$
$$a_1^{-2} a_2^{-2} \ldots a_{n-1}^{-1} a_n^{2} = 1,$$
$$\ldots$$
$$a_1^{2} a_2^{2} \ldots a_{n-1}^{2} a_n = 1.$$

These are the same relations as those obtained in Theorem 4.2.13 and so

$$|\underset{\sim}{H}(r+1,\ 2n,\ r)/\underset{\sim}{H}'(r+1,\ 2n,\ r)| = |\underset{\sim}{H}(r,\ 2n,\ r-1)/\underset{\sim}{H}'(r,\ 2n,\ r-1)|.$$

Lemma 4.2.16. If $r \equiv n \bmod 2n$ where $n = 2,3$ then $H'(r+1,\ 2n,\ r)$ is the trivial group.

Proof. The proof is similar to that of Lemma 4.2.14. We consider first the case when $r \equiv 2 \bmod 4$ and let $r = 4u + 2$. Then $H(r+1,\ 2n,\ r)$ has a presentation on the four generators a_1, a_2, a_3, a_4 subject to the four relations

$$(a_1 a_2 a_3 a_4)^u a_1 a_2 a_3 = a_4 a_1 (a_2 a_3 a_4 a_1)^u, \qquad (1)$$

$$(a_2 a_3 a_4 a_1)^u a_2 a_3 a_4 = a_1 a_2 (a_3 a_4 a_1 a_2)^u, \qquad (2)$$

$$(a_3 a_4 a_1 a_2)^u a_3 a_4 a_1 = a_2 a_3 (a_4 a_1 a_2 a_3)^u, \qquad (3)$$

$$(a_4 a_1 a_2 a_3)^u a_4 a_1 a_2 = a_3 a_4 (a_1 a_2 a_3 a_4)^u. \qquad (4)$$

We know by Lemma 4.2.15 that

$$a_4 = a_2^{-1}, \qquad a_3 = a_1^{-1}. \tag{5}$$

From (1)

$$(a_1a_2a_3a_4)^u a_1a_2a_3 = a_4a_1(a_2a_3a_4a_1)^u, \tag{6}$$

from (2) and (6)

$$(a_2a_3a_4a_1)^u a_2a_3a_4a_3 = a_4a_1(a_2a_3a_4a_1)^u, \tag{7}$$

from (3) and (7)

$$(a_3a_4a_1a_2)^u a_3a_4a_1a_4a_3 = a_4a_1(a_2a_3a_4a_1)^u, \tag{8}$$

and from (4) and (8)

$$(a_4a_1a_2a_3)^u a_4a_1a_2a_1a_4a_3 = a_4a_1(a_2a_3a_4a_1),$$

that is,

$$a_2a_1a_4a_3 = 1, \tag{9}$$

from which, as in Lemma 4.2.14, the result follows.

Now consider the case when $r \equiv 3 \bmod 6$ and let $r = 6v + 3$. Then $\underset{\sim}{H}(r + 1, 6, r)$ has a presentation on the six generators a_1, a_2, a_3, a_4, a_5, a_6 subject to the six relations

$$(a_1a_2a_3a_4a_5a_6)^v a_1a_2a_3a_4 = a_5a_6a_1(a_2a_3a_4a_5a_6a_1)^v, \tag{10}$$

$$(a_2a_3a_4a_5a_6a_1)^v a_2a_3a_4a_5 = a_6a_1a_2(a_3a_4a_5a_6a_1a_2)^v, \tag{11}$$

$$(a_3a_4a_5a_6a_1a_2)^v a_3a_4a_5a_6 = a_1a_2a_3(a_4a_5a_6a_1a_2a_3)^v, \tag{12}$$

$$(a_4a_5a_6a_1a_2a_3)^v a_4a_5a_6a_1 = a_2a_3a_4(a_5a_6a_1a_2a_3a_4)^v, \tag{13}$$

$$(a_5a_6a_1a_2a_3a_4)^v a_5a_6a_1a_2 = a_3a_4a_5(a_6a_1a_2a_3a_4a_5)^v, \tag{14}$$

$$(a_6a_1a_2a_3a_4a_5)^v a_6a_1a_2a_3 = a_4a_5a_6(a_1a_2a_3a_4a_5a_6)^v. \tag{15}$$

From (10),

$$a_1a_2a_3(a_4a_5a_6a_1a_2a_3)^v a_4 = a_5a_6a_1(a_2a_3a_4a_5a_6a_1)^v, \tag{16}$$

from (12)

$$a_3a_4a_5(a_6a_1a_2a_3a_4a_5)^v a_6a_4 = a_5a_6a_1(a_2a_3a_4a_5a_6a_1)^v, \tag{17}$$

from (14)

$$a_5a_6a_1(a_2a_3a_4a_5a_6a_1)^v a_2a_6a_4 = a_5a_6a_1(a_2a_3a_4a_5a_6a_1)^v,$$

that is

$$a_2a_6a_4 = 1. \tag{18}$$

Similarly, from (11), (13), (15)

$$a_5a_3a_1 = 1. \tag{19}$$

By Lemma 4.2.15,

$$a_6 = a_3^{-1}, \quad a_5 = a_2^{-1}, \quad a_4 = a_1^{-1}. \tag{20}$$

From (18), (19), (20) it follows as in Lemma 4.2.14 that $\underset{\sim}{H}(r+1, 6, r)$ is abelian and so $\underset{\sim}{H}'(r+1, 6, r)$ is the trivial group.

<u>Theorem 4.2.17.</u> (i) If $r \equiv 2,3 \bmod 4$ then $\underset{\sim}{H}(r, 4, r-1) \cong \mathbb{Z}_5$.

(ii) If $r \equiv 3,4 \bmod 6$ then $\underset{\sim}{H}(r, 6, r-1) \cong \mathbb{Z}_{13}$.

<u>Proof.</u> When $r \equiv 2 \bmod 4$, $|\underset{\sim}{H}(r, 4, r-1)/\underset{\sim}{H}'(r, 4, r-1)| = 5$ by Theorem 4.2.13 and $\underset{\sim}{H}'(r, 4, r-1) = 1$ by Lemma 4.2.14. Therefore $\underset{\sim}{H}(r, 4, r-1) \cong \mathbb{Z}_5$.

By Lemma 4.2.15 $|\underset{\sim}{H}(r+1, 4, r)/\underset{\sim}{H}'(r+1, 4, r)| = 5$ and by Lemma 4.2.16 $\underset{\sim}{H}'(r+1, 4, r) = 1$ when $r \equiv 3 \bmod 4$. Therefore $\underset{\sim}{H}(r, 4, r-1) \cong \mathbb{Z}_5$ when $r \equiv 3 \bmod 4$.

Similarly from Theorem 4.2.13 $|\underset{\sim}{H}(r, 6, r-1)/\underset{\sim}{H}'(r, 6, r-1)| = 13$ when $r \equiv 3 \bmod 6$ and again using Lemmas 4.2.14, 4.2.15 and 4.2.16, $\underset{\sim}{H}(r, 6, r-1) \cong \mathbb{Z}_{13}$ when $r \equiv 3,4 \bmod 6$.

4.3 Groups of order 1512

In this section we shall show that $F(3, 6)$ and $F(3, 6, 5, 2)$ are two non-isomorphic groups of order 1512, and that neither is isomorphic to the group $G(2, 3, -2)$ discussed in 3.4.6. Further we shall show that they are not metacyclic and so are cyclically presented finite non-metacyclic groups of deficiency zero. $F(3, 6)$ was the first, and as far as we know is still the only, finite non-metacyclic Fibonacci group to be discovered.

$F(3, 6)$. $F(3, 6) \cong \langle a,b,c\ d,e,f \mid abc = d,\ bcd = e,\ cde = f,\ def = a,\ efa = b,\ fab = c \rangle$. We find a 2-generator 2-relation presentation for $F(3, 6)$.
From $def = a$ and $efa = b$,

$$d = a^2b^{-1}, \tag{1}$$

from $abc = d$ and (1)

$$c = b^{-1}ab^{-1}, \tag{2}$$

from $bcd = e$, (1) and (2)

$$e = ab^{-1}a^2b^{-1}, \tag{3}$$

from $cde = f$, (1), (2) and (3)

$$f = b^{-1}ab^{-1}a^2b^{-1}ab^{-1}a^2b^{-1}. \tag{4}$$

Therefore $F(3, 6)$ may be reduced to a 2-generator group. The relations for $F(3, 6)$ are $efa = b$ and $fab = c$ where c, e and f are given by (2), (3) and (4) respectively. Thus

$$F(3, 6) \cong \langle a,b \mid a^2b^{-1}ab^{-1}a^2b^{-1}ab = 1,\ ab^{-1}a^2b^{-2}ab^{-3} = 1 \rangle .$$

Coset enumeration gives immediately that $|F(3, 6)| = 1512 = 2^3 3^3 7$ and that a and b have order 84. A Sylow 7-subgroup is therefore generated by a^{12} and since the homomorphic image obtained by adding $[a^{12}, b] = 1$ also has order 1512, there is a unique Sylow 7- subgroup in the centre of $F(3, 6)$.

The elements a^{28}, a^2b^{18} and $a^4b^{-1}a^2b^{-1}$ are three elements of order 3 and they generate a subgroup of order 27. From hand calculations it was shown in (13) that this Sylow 3-subgroup was $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. However I have now found an error in the calculation and the Sylow 3-subgroup is in fact not $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. If we add to the relations for $\underset{\sim}{F}(3, 6)$, the relations that a^{28}, a^2b^{18} and $a^4b^{-1}a^2b^{-1}$ commute, namely $a^{28}b^{18}a^{-28}b^{-18} = 1$, $a^{28}b^{-1}a^2b^{-1}a^{-28}ba^{-2}b = 1$ and $b^{18}a^4b^{-1}a^2b^{-19}a^{-2}ba^{-2}ba^{-2} = 1$ we obtain a group of order 504 and so the Sylow 3-subgroup is non-abelian. The coset enumeration programme (1) shows that the Sylow 3-subgroup may be generated by a^2b^{18} and $a^4b^{-1}a^2b^{-1}$. It is the non-abelian group of order 27 and exponent three given on page 135 of (24) by the presentation $\langle R,S \mid R^3 = S^3 = (RS)^3 = (R^{-1}S)^3 = 1 \rangle$ for if we add to the 2-generator 2-relation presentation for $\underset{\sim}{F}(3, 6)$, that is

$$\underset{\sim}{F}(3, 6) \cong \langle a,b \mid a^2b^{-1}ab^{-1}a^2b^{-1}ab = 1,\ ab^{-1}a^2b^{-2}ab^{-3} = 1 \rangle$$

the relations $(a^2b^{18})^3 = 1$, $(a^4b^{-1}a^2b^{-1})^3 = 1$, $(a^2b^{18}a^4b^{-1}a^2b^{-1})^3 = 1$ and $(a^2b^{-1}a^2b^{-19})^3 = 1$ we still obtain a group of order 1512. Hence a^2b^{18} and $a^4b^{-1}a^2b^{-1}$ generate a homomorphic image of the group given by $\langle R,S \mid R^3 = S^3 = (RS)^3 = (R^{-1}S)^3 = 1 \rangle$ but since they generate a group of order 27, the Sylow 3-subgroup is the non-abelian group of order 27 and exponent three. By considering the normaliser in $\underset{\sim}{F}(3, 6)$ of this subgroup and showing that the index of the normaliser is one, the Sylow 3-subgroup is normal. A Sylow 2-subgroup is generated by a^{21} and $(b^{-1}ab^{-1}a^2b^{-1})^{21}$ two elements of order 4. These two generators do not commute and so the Sylow 2-subgroup is necessarily the quaternion group Q_8 of order 8 since it is generated by two elements of order 4. Since $(b^{-1}ab^{-1}a^2b^{-1})^{21} \in N_{\underset{\sim}{F}(3,\ 6)}\langle a^{21} \rangle$ and since $|\underset{\sim}{F}(3, 6) : N_{\underset{\sim}{F}(3,\ 6)}(\langle a^{21}, (b^{-1}ab^{-1}a^2b^{-1})^{21} \rangle)| = 9$, there are 9 Sylow 2-subgroups each having normaliser $Q_8 \times \mathbb{Z}_{21}$ where $\mathbb{Z}_{21}$ lies in the centre of $\underset{\sim}{F}(3, 6)$. Although $\underset{\sim}{F}(3, 6)$ is obviously not nilpotent it is however soluble. From the 2-generator 2-relation presentation we have, if we abelianise the relations, $a^6 = b^2$ and $a^4 = b^6$. Therefore $|\underset{\sim}{F}(3, 6) : \underset{\sim}{F}'(3, 6)| = 28$ and $|\underset{\sim}{F}'(3, 6)| = 54 = 2.3^3$ which, being of the form $p^\alpha q^\beta$ where p and q are primes, is soluble. Therefore $\underset{\sim}{F}(3, 6)$ is soluble.

<u>$F(3, 6, 5, 2)$</u> $\quad \underset{\sim}{F}(3, 6, 5, 2) \cong \langle a, b, c, d, e, f \mid ace = d, bdf = e, cea = f, dfb = a, eac = b, fbd = c \rangle$. We first find a 2-generator 2-relation presentation for $\underset{\sim}{F}(3, 6, 5, 2)$. From cea = f we may eliminate f to obtain the five relations

$$ace = d, \tag{1}$$

$$bdcea = e, \tag{2}$$

$$dceab = a, \tag{3}$$

$$eac = b, \tag{4}$$

$$eabd = 1. \tag{5}$$

From (2) and (3)

$$e = bab^{-1}, \tag{6}$$

from (5) and (6)

$$d = b^{-1}a^{-1}ba^{-1}b^{-1}, \tag{7}$$

from (4) and (6)

$$c = a^{-1}ba^{-1}. \tag{8}$$

The five relations are therefore reduced to the two relations (1) and (3) where c, d and e are given in terms of a and b by (6), (7) and (8). The relations (1) and (3) are thus $ba^{-1}bab^{-1} = b^{-1}a^{-1}ba^{-1}b^{-1}$ and $b^{-1}a^{-1}ba^{-1}b^{-1}a^{-1}ba^{-1}bab^{-1}ab = a$. Therefore

$$\underset{\sim}{F}(3, 6, 5, 2) \cong \langle a,b \mid a^{-1}ba^2b^{-1}ab^2 = 1,\ (ba^{-1}b^{-1}a^{-1})^2ba^{-1}bab^{-1}a = 1 \rangle.$$

The coset enumeration programme (1) shows that $|\underset{\sim}{F}(3, 6, 5, 2)| = 1512 = 2^3 3^3 7$. The element a has order 8 and so a Sylow 2-subgroup is cyclic and generated by a. It is not normal. The elements $[a, b^2]$ and $[a^{-2}, b^{-1}]$ are both of order 3, and since the subgroup $\langle [a, b^2], [a^{-2}, b^{-1}] \rangle$ is of index 56 in $\underset{\sim}{F}(3, 6, 5, 2)$ they generate a Sylow 3-subgroup. This subgroup is normal but is not abelian, the Sylow 3-subgroup being, as in the case of $\underset{\sim}{F}(3, 6)$ the non-abelian group of order 27 and exponent three.

Since $|\underset{\sim}{F}(3, 6, 5, 2) : \underset{\sim}{F}'(3, 6, 5, 2)| = 8$, $|\underset{\sim}{F}'(3, 6, 5, 2)| = 189$, which being of the form $p^{\alpha}q^{\beta}$ where p and q are primes, is soluble. Therefore $\underset{\sim}{F}(3, 6, 5, 2)$ is soluble. It is of course not abelian.

$\underset{\sim}{F}(3, 6, 5, 2)$ is not isomorphic to $\underset{\sim}{F}(3, 6)$ since, for example, $\underset{\sim}{F}(3, 6)$ has Q_8 as a Sylow 2-subgroup whereas $\underset{\sim}{F}(3, 6, 5, 2)$ has $\mathbb{Z}_8$ as a Sylow 2-subgroup. Further since $\underset{\sim}{G}(2, 3, -2)$, see 3.4.6, has $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ as a Sylow 2-subgroup it is isomorphic to neither $\underset{\sim}{F}(3, 6)$ nor $\underset{\sim}{F}(3, 6, 5, 2)$.

4.4. Results from computing.

In this section we give details of groups whose orders have been determined by coset enumeration. As already mentioned I had at an early stage in looking at Fibonacci groups found that $\underset{\sim}{F}$ (3,6) was a group of order 1512. The structure of this group and the fact that it is not metacyclic have been discussed in 4.3. Coset enumeration also enables us to determine that $\underset{\sim}{F}$ (6,4) is a group of order 125.

In addition while still considering the Fibonacci groups $\underset{\sim}{F}(r,n)$ we have shown, using coset enumeration, that the addition of the relation $a_1 = a_6$ to $\underset{\sim}{F}(2,10)$ yields a metacyclic group of order 253, thus answering a question posed in (31). This question is further discussed by Brunner (5) who shows that $\underset{\sim}{F}$ (2,10) is infinite.

We have made a systematic computer study of the groups $\underset{\sim}{H}(r,n,s)$ for $s \leq 6$, $r,n \leq 7$ and have found that the orders obtained were all consequences of the lemmas and theorems of 4.2 and 4.6. The only other information obtained about the groups $\underset{\sim}{H}(r,n,s)$ was that $\underset{\sim}{H}$ (9,4,2) is a group of order 1015. Now consider the groups $\underset{\sim}{F}(r,n,k,h), h \neq 1$. E.F. Robertson and I, and D.L. Johnson and H. Mawdesley, had independently investigated the groups $\underset{\sim}{F}(2,n,k,h)$. The work of Johnson and Mawdesley is given in (30) and as a table of the groups $\underset{\sim}{F}(2,n,k,h)$, $n,k,h \leq 6$ is given there we need not reproduce it here. The group of most interest on the table is perhaps $\underset{\sim}{F}(2,6,2,3)$ a group of order 56. We have shown further that $\underset{\sim}{F}(3,6,1,3)$, $\underset{\sim}{F}(3,6,1,4)$ and $\underset{\sim}{F}(3,6,3,4)$ are groups of order 728,1512 and 1512 respectively.

In fact we may use Lemma 4.2.1 to show that the latter two groups are both isomorphic to the non-metacyclic group $\underset{\sim}{F}(3,6,5,2)$ discussed in 4.3.

Next we show that the groups $\underset{\sim}{F}(3,6,2,3)$, $\underset{\sim}{F}(3,6,3,3)$ and $\underset{\sim}{F}(3,6,4,2)$ are infinite.

$\underline{\underset{\sim}{F}(3,6,2,3)}$. $F(3,6,2,3) \cong \langle a,b,c,d,e,f \mid ada=c, beb=d, cfc=e, dad=f, ebe=a, fcf=b \rangle$.

We use relations (1), (2), (4) and (5) to get

$$f = dad,$$
$$d = beb,$$
$$b = e^{-1} a\, e^{-1},$$
$$c = ada,$$

that is

$$b = e^{-1} a\, e^{-1},$$
$$c = ae^{-1}ae^{-1}ae^{-1}a,$$
$$d = e^{-1}ae^{-1}ae^{-1},$$
$$f = (e^{-1}a)^5\, e^{-1}.$$

From relations (3) and (6),

$$\underset{\sim}{F}(3,6,2,3) \cong \langle a,e \mid (ae^{-1})^{13} = 1 \rangle .$$

Therefore $\underset{\sim}{F}(3,6,2,3)$ is infinite.

$\underline{\underset{\sim}{F}(3,6,3,3)}$. $F(3,6,3,3) \cong \langle a,b,c,d,e,f \mid ada=d, beb=e, cfc=f, dad=a, ebe=b, fcf=c \rangle$.

Since $\underset{\sim}{F}(3,6,3,3)/\underset{\sim}{F}'(3,6,3,3)$ is generated by six elements it follows by Theorem 1.3.3 that $\underset{\sim}{F}(3,6,3,3)$ is infinite.

$\underline{\underset{\sim}{F}(3,6,4,2)}$. $\underset{\sim}{F}(3,6,4,2) \cong \langle a,b,c,d,e,f \mid ace=c, bdf=d, cea=e, dfb=f, eac=a, fbd=b \rangle$.

From relation (1) $e = c^{-1}a^{-1}c$, and from relation (2) $f = d^{-1}b^{-1}d$.

Therefore

$$F \cong \langle a,c,b,d \mid aca=cac, cac^{-1}=ac^{-1}a^{-1}, bdb=dbd, dbd^{-1}=bd^{-1}b^{-1} \rangle,$$

which may be written as

$$F \cong G * G$$

where $G \cong \langle a,c \mid aca = cac, cac^{-1} = ac^{-1}a^{-1} \rangle$. Thus $F(3,6,4,2)$ is infinite since $G/G' \cong \mathbb{Z}_2$ showing that G is non-trivial.

We now look at the generalised Fibonacci groups $F(r,n,k)$. Coset enumeration gives immediately that $F(2,6,2)$ is $\mathbb{Z}_7$, $F(6,3,2)$ is $\mathbb{Z}_5$ and that $F(4,4,2)$ has order 39. Apart from the non-metacyclic generalised Fibonacci groups isomorphic to $F(3,6)$ and those finite non-metacyclic groups, for example $F(2,5,2)$, arising from Theorem 4.2.10 another finite non-metacyclic generalised Fibonacci group is $F(3,3,2)$. Coset enumeration gives that the order of $F(3,3,2)$ is 48 and also that it has S_3 as a homomorphic image. ~~and therefore~~ It is not metacyclic. Further the derived group of $F(3,3,2)$ is the binary tetrahedral group $\langle 2,3,3 \rangle$, see 1.4.

$$F(3,3,2) \cong \langle a,b,c \mid abc = b, bca = c, cab = a \rangle,$$

which may be written as

$$F(3,3,2) \cong \langle a,b \mid aba = bab, bab^{-1} = ab^{-1}a^{-1} \rangle,$$

The subgroup generated by $ab = x$ and $ba = y$ has index 2 in $F(3,3,2)$ and since $|F(3,3,2)/F'(3,3,2)| = 2$, the derived group is generated by ab and ba. The Todd-Coxeter algorithm gives as a presentation for the derived group,

$$\langle x,y \mid yxy = x^2, xyx = y^2 \rangle,$$

and this is a presentation for $\langle 2,3,3 \rangle$, the binary tetrahedral group.

We next show that $\mathbf{F}(3,5,4)$, $\mathbf{F}(3,6,5)$ and $\mathbf{F}(5,5,3)$ are infinite.

$\underline{\mathbf{F}(3,5,4)}$. $\mathbf{F}(3,5,4) \cong \langle a,b,c,d,e \mid abc = b, bcd = c, cde = d, dea = e, eab = a \rangle$.

From relation (5),

$$e = ab^{-1}a^{-1}, \qquad \text{(A)}$$

from relations (4) and (A),

$$d = ab^{-1}a^{-1}ba^{-1}, \qquad \text{(B)}$$

and from relations (3), (A) and (B),

$$c = ab^{-1}a^{-1}bab\,a^{-1}. \qquad \text{(C)}$$

Therefore $\mathbf{F}(3,5,4)$ has a 2-generator 2-relation presentation

$$\langle a,b \mid abab^{-1}a^{-1}baba^{-1}b^{-1} = 1,\ aba^{-1}b^{-1}ab^{-1}a^{-1}bab^{-1} = 1 \rangle.$$

The subgroup $H = \langle a,b^2 \rangle$ has index 10 in $\mathbf{F}(3,5,4)$. With $x = a, y = b^2$ we may use the Todd-Coxeter algorithm and the programme CCRG, see 2.5, to show that

$$H \cong \langle x,y \mid yxy^{-1}x^{-1}yxy^{-1}x^{-1}y^{-1}xyx^{-1}y^{-1}xyx^{-1} = 1,$$
$$y^{-1}x^{-1}yx^{-1}y^{-1}xyx^{-1}y^{-1}xyxy^{-1}x^{-1}yx = 1 \rangle.$$

From this presentation it is immediate that $H/H' \cong \mathbb{Z} \times \mathbb{Z}$ and so $\mathbf{F}(3,5,4)$ is infinite.

$\underline{\mathbf{F}(3,6,5)}$. $\mathbf{F}(3,6,5) \cong \langle a,b,c,d,e,f \mid abc=b, bcd=c, cde=d, def=e, efa=f, fab=a \rangle$.

From relations (6), (5), (4) and (3) respectively we obtain

$$f = ab^{-1}a^{-1},$$

$$e = ab^{-1}a^{-1}ba^{-1},$$

$$d = ab^{-1}a^{-1}baba^{-1},$$

$$c = ab^{-1}a^{-1}bab^{-1}aba^{-1}.$$

Therefore $\underset{\sim}{F}(3,6,5)$ has a 2-generator 2-relation presentation

$$\langle a,b \mid abab^{-1}a^{-1}bab^{-1}aba^{-1}b^{-1} = 1,\ ab^{-1}a^{-1}bab^{-1}a^{-1}b^{-1}aba^{-1}b^{-1} = 1 \rangle.$$

The subgroup $H = \langle a^2, b^2, a^{-1}b^2a, b^{-1}a^2b \rangle$ has index 24 in $\underset{\sim}{F}(3,6,5)$.

With $x = a^2, y = b^2, z = a^{-1}b^2a, t = b^{-1}a^2b$,

$$H \cong \langle x,y,z,t \mid t^{-1}y^{-1}xzx^{-1}tz^{-1}yt^{-1}xz^{-1}x^{-1}yty^{-1}z = 1,$$

$$xzx^{-1}yz^{-1}yt^{-1}y^{-1}xzx^{-1}ty^{-1}zt^{-1}xz^{-1}x^{-1}yty^{-1}tz^{-1}yt^{-1} = 1,$$

$$y^{-1}xzy^{-1}tz^{-1}t^{-1}xz^{-1}x^{-1}yty^{-1}zyt^{-1} = 1,$$

$$y^{-1}xzx^{-1}tz^{-1}x^{-1}yty^{-1}zt^{-1}xz^{-1}yt^{-1} = 1,$$

$$y^{-1}xzx^{-1}t^2y^{-1}zt^{-1}xz^{-1}x^{-1}yz^{-1}yt^{-1} = 1 \rangle.$$

$H/H' \cong \langle x,y,z,t \mid y = 1, x = 1 \rangle$ and so, since H has an infinite homorphic image, $\underset{\sim}{F}(3,6,5)$ is infinite.

$\underline{\underset{\sim}{F}(5,5,3)}$. $F(5,5,3) = \langle a,b,c,d,e \mid abcde = c,\ bcdea = d,\ cdeab = e,$
$deabc = a,\ eabcd = b \rangle.$

This presentation is equivalent to the presentation

$$\langle a,b,c,d,e \mid cdeab = e,\ a^{-1}ca = d,\ b^{-1}db = e,\ c^{-1}ec = a,\ d^{-1}ad = b \rangle.$$

From relations (4), (2) and (5) respectively we obtain,

$$e = cac^{-1},$$

$$d = a^{-1}ca,$$

and

$$b = a^{-1}c^{-1}aca.$$

Therefore $\underset{\sim}{F}(5,5,3)$ has a 2-generator 2-relation presentation

$$\langle a,c \mid cacac = acaca,\ acaca^{-2}c\ acac^{-2} = 1 \rangle .$$

The subgroup $H = \langle (ac)^2, (ca)^2 \rangle$ has index 4. With $x = (ac)^2$, $y = (ca)^2$,

$$H \cong \langle x,y \mid x^5 = y^5 = (xy)^3 \rangle .$$

This is ~~the~~ a binary polyhedral group $\langle 5,5,3 \rangle$ which is shown in (24) to be infinite; see also section 1.4 of this thesis. It may be interesting to note that $\underset{\sim}{F}'(3,3,2)$ is $\langle a_1a_2, a_2a_1 \rangle$ which we have shown is the binary tetrahedral group $\langle 3,3,2 \rangle$, while $\underset{\sim}{F}'(5,5,3)$ is $\langle (a_1a_2)^2, (a_2a_1)^2 \rangle$ which is the binary polyhedral group $\langle 5,5,3 \rangle$. In general it is easy to show that $H_n = \langle (a_1a_2)^n, (a_2a_1)^n \rangle$ is a subgroup of $\underset{\sim}{F}'(2n+1, 2n+1, n+1)$ but we have been unable to determine whether H_n is the binary polyhedral group $\langle 2n+1, 2n+1, n+1 \rangle$ for $n \geq 3$.

We conclude this section by summarising our knowledge of the generalised Fibonacci groups $\underset{\sim}{F}(r,n,k)$ for $r,n \leq 6$. We give first the table from (31) where the results for $k = 1$, including our own results for $k = 1$, are tabulated. The fact that $\underset{\sim}{F}(2,7)$ is $\mathbb{Z}_{29}$, see (5), has been included in this table. The second table gives the orders of the generalised Fibonacci groups for $k \neq 1$. In the case of free groups, cyclic groups and SL(2,5) their isomorphism type is indicated. Other than these all the finite groups are metacyclic except for $\underset{\sim}{F}(3,3,2)$ and $\underset{\sim}{F}(3,6,3)$.

r \ n	1	2	3	4	5	6	7
1	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
2	$\mathbb{Z}_1$	$\mathbb{Z}_1$	Q	$\mathbb{Z}_5$	$\mathbb{Z}_{11}$	∞	$\mathbb{Z}_{29}$
3	$\mathbb{Z}_2$	Q	$\mathbb{Z}_2$	∞	$\mathbb{Z}_{22}$	1512	
4	$\mathbb{Z}_3$	$\mathbb{Z}_3$	63	$\mathbb{Z}_3$	∞		
5	$\mathbb{Z}_4$	24	∞	624	$\mathbb{Z}_4$	∞	
6	$\mathbb{Z}_5$	$\mathbb{Z}_5$	$\mathbb{Z}_5$	125	7775	$\mathbb{Z}_5$	∞
7	$\mathbb{Z}_6$	48	342	∞		117648	$\mathbb{Z}_6$

Fibonacci groups.

k	2					3				4			5		6
r \ n	2	3	4	5	6	3	4	5	6	4	5	6	5	6	6
1	F_2	$\mathbb{Z}$	F_2	$\mathbb{Z}$	F_2	F_3	$\mathbb{Z}$	$\mathbb{Z}$	F_3	F_4	$\mathbb{Z}$	F_2	F_5	$\mathbb{Z}$	F_6
2	1	1	$\mathbb{Z}_5$	SL(2,5)	$\mathbb{Z}_7$	1	1	$\mathbb{Z}_{11}$	$\mathbb{Z}_7$	1	1	∞	1	1	1
3	∞	48	∞	$\mathbb{Z}_{22}$	∞	$\mathbb{Z}_2$	∞	$\mathbb{Z}_2$	1512	∞	∞	∞	$\mathbb{Z}_2$	∞	∞
4	$\mathbb{Z}_3$	63	39	$\mathbb{Z}_3$		∞	39		∞	$\mathbb{Z}_3$			$\mathbb{Z}_3$		∞
5	∞	$\mathbb{Z}_4$	∞		∞	$\mathbb{Z}_4$	624	∞		∞		∞	$\mathbb{Z}_4$		∞
6	$\mathbb{Z}_5$	$\mathbb{Z}_5$	125	7775		$\mathbb{Z}_5$	$\mathbb{Z}_5$	7775		$\mathbb{Z}_5$	7775:		∞		$\mathbb{Z}_5$

Generalised Fibonacci groups.

4.5 Metacyclic Fibonacci groups

In this section we show that if $r \equiv 1 \bmod n$ and $(k, n) = 1$, then $\underset{\sim}{F}(r, n, k)$ is metacyclic of order $r^n - 1$ and has a 2-generator 2-relation presentation. We show first how the modified Todd-Coxeter algorithm, see 2.2, may be used to find a 2-generator 3-relation presentation for $\underset{\sim}{F}(r, n, k)$ and also to obtain the order $r^n - 1$. Since the appearance of our work in (10) and (11) it has been suggested by A. M. Brunner in a private communication that there are other and perhaps shorter methods of obtaining the order $r^n - 1$. However our use of the modified Todd-Coxeter algorithm enables us to find both the order of $\underset{\sim}{F}(r, n, k)$ and, in Theorem 4.5.2, a 2-generator 2-relation presentation for $\underset{\sim}{F}(r, n, k)$. As mentioned in the introduction we thus include a solution to problem (3) of (31), namely that of finding in the case where $r \equiv 1 \bmod n$, 2-generator 2-relation presentations for, and the orders of, the Fibonacci groups. Although the results that follow for generalised Fibonacci groups are special cases of the results of section 4.6 they are included for three reasons.

Firstly because of the complexity of the proofs for the Fibonacci-type groups in section 4.6 the proofs in this section are a useful introduction to that section. Secondly we are able to determine in Corollary 3 to Theorem 4.5.1 an isomorphism between Fibonacci groups and generalised Fibonacci groups. Thirdly we obtain in Theorem 4.5.2 a much more agreeable 2-generator 2-relation presentation for the groups $\underset{\sim}{F}(r, n, k)$ than is otherwise obtained as a special case of Theorem 4.6.1.

<u>Theorem 4.5.1.</u> Suppose $r \equiv 1 \bmod n$ and $(k, n) = 1$ then $\underset{\sim}{F}(r, n, k)$ is metacyclic of order $r^n - 1$. Further

$$\underset{\sim}{F}(r, n, k) \cong \langle x, y \mid y^{-1}xy = x^{r^h},\ y^n = x^{(r^n-1)/(n(r-1))},\ x^{(r^n-1)/n} = 1 \rangle,$$

where $hk \equiv 1 \bmod n$ and $1 \leq h \leq n-1$.

<u>Proof.</u> Denote by x the element $a_1a_2 \ldots a_n$ of $\underset{\sim}{F}(r, n, k)$ and let $H = \langle x \rangle$. We use the modified Todd-Coxeter algorithm to find the index of H in $\underset{\sim}{F}(r, n, k)$ and to find a presentation for H. Further we show that if $y = a_{1+k}$ then x and y together generate $\underset{\sim}{F}(r, n, k)$. Again we use the modified Todd-Coxeter algorithm, this time to find a presentation for $\underset{\sim}{F}(r, n, k)$. The notation is as in section 2.4.

Since $r \equiv 1 \bmod n$, let $r = nt + 1$. Define n cosets of H as follows: let coset $1 = H$ and for $2 \leq i \leq n$ define coset i by $i = (i - 1)a_{i-1}$. Then the subgroup generator gives us $na_n = x\,1$. Define $b_o = 0$, $b_1 = t$ and, inductively, $b_j = (nt + 1)b_{j-1} + t$.

Using relation (1) we obtain $1\,a_{k+1} = x^{b_1}2$ and, in general, from relation (j) we obtain

$$j\,a_{k+j} = x^{b_1}(j + 1), \qquad 1 \leq j \leq n - 1.$$

Also relation (n) being $(a_na_1 \ldots a_{n-1})^ta_n = a_k$ gives $na_k = x^{b_1+1}.1$. From relation $(k + 1)$, that is the relation $(a_{k+1}\,a_{k+2} \ldots a_k)^ta_{k+1} = a_{2k+1}$, we obtain the information $1a_{2k+1} = 1(a_{k+1}\,a_{k+2} \ldots a_k)^ta_{k+1} = x^{ntb_1+t}\cancel{x}x^{b_1}2 = x^{b_2}2$. Proceeding successively in this way we obtain from relation $((\alpha - 1)k + i)$

the information

$$i.a_{\alpha k+i} = \begin{cases} x^{b_\alpha}.(i+1), & 1 \le i \le n-1, \\ x^{b_\alpha+1}.1, & i = n, \end{cases}$$

and since k is coprime to n the coset enumeration terminates showing that the index of H in $\underset{\sim}{F}(r, n, k)$ is n. In addition H is normal in $\underset{\sim}{F}(r, n, k)$ since $ix = xi$, $1 \le i \le n$. As in 2.2 the relations for the subgroup H are obtained from the n equations

$$i.(a_{i-k}\, a_{i+1-k} \ldots a_{i-1-k})^t a_{i-k} = i.a_i, \qquad 1 \le i \le n. \qquad (*)$$

Now $$i.a_i = \begin{cases} i + 1, & 1 \le i \le n-1, \\ x \,.\, 1, & i = n, \end{cases}$$

and $$i.a_{i-k} = \begin{cases} x^{b_{n-1}}.(i + 1), & 1 \le i \le n-1, \\ x^{b_{n-1}+1}.1, & i = n. \end{cases}$$

Therefore for $1 \le i \le n-1$ the equations (*) give

$$x^{(nb_{n-1} + 1)t}\, x^{b_{n-1}}.(i + 1) = (i + 1),$$

that is

$$x^{b_{n-1}(1+nt) + t} = 1,$$

$$\text{or} \qquad x^{b_n} = 1.$$

When $i = n$, the nth equation of (*) gives

$$x^{(nb_{n-1}+1)t}\, x^{b_{n-1}+1}.1 = x.1,$$

and again this reduces to

$$x^{b_n} = 1.$$

Thus from each of the equations (*) we have obtained the relation $x^{b_n} = 1$ and so $H \cong \langle x | x^{b_n} = 1 \rangle$.

Assume by induction that $b_{j-1} = \sum_{s=1}^{j-1} \binom{j-1}{s} n^{s-1} t^s$. Then

$$\begin{aligned}
b_j &= (nt + 1)b_{j-1} + t, \\
&= (nt + 1) \sum_{s=1}^{j-1} \binom{j-1}{s} n^{s-1}\, t^s + t, \\
&= \sum_{s=1}^{j-1} \binom{j-1}{s} n^s t^{s+1} + \sum_{s=1}^{j-1} \binom{j-1}{s} n^{s-1} t^s + t, \\
&= n^{j-1} t^j + \sum_{s=1}^{j-2} \binom{j-1}{s} n^s t^{s+1} + \sum_{s=2}^{j-1} \binom{j-1}{s} n^{s-1} t^s + (j-1)t + t, \\
&= n^{j-1} t^j + \sum_{s=2}^{j-1} \left\{ \binom{j-1}{s-1} + \binom{j-1}{s} \right\} n^{s-1} t^s + jt, \\
&= \sum_{s=1}^{j} \binom{j}{s}\, n^{s-1} t^s \quad \text{since} \quad \binom{j-1}{s-1} + \binom{j-1}{s} = \binom{j}{s}.
\end{aligned}$$

Therefore the order of $\underset{\sim}{F}(r, n, k)$ is $n\, b_n$ where

$$\begin{aligned}
n\, b_n &= n \sum_{s=1}^{n} \binom{n}{s}\, n^{s-1}\, t^s, \\
&= \sum_{s=0}^{n} \binom{n}{s}\, (n\, t)^s - 1, \\
&= (nt + 1)^n - 1, \\
&= r^n - 1.
\end{aligned}$$

$\underset{\sim}{F}(r, n, k)$ is therefore metacyclic of order $r^n - 1$.

For the second part of the theorem we show that if $y = a_{1+k}$, then x and y together generate $\underset{\sim}{F}(r, n, k)$. We use the modified Todd-Coxeter algorithm as before but this time from the subgroup generator y we have the additional information $1a_{1+k} = y1$. We follow through the collapses that occur as a consequence of this information and show that in fact we get complete collapse. This will imply that x and y generate $\underset{\sim}{F}(r, n, k)$.

From relation (1) we obtain $1a_{1+k} = x^t2$ and so $2 = x^{-t}y1$. But $1a_{\alpha k+1} = x^{b_\alpha}2 = x^{b_\alpha - t}y1$. Since k is coprime to n,

$$1a_i = x^{b_{h(i-1)} - t}y1, \qquad 1 \leq i \leq n,$$

where $hk \equiv 1 \bmod n$, $1 \leq h \leq n-1$.

Let K be the subgroup generated by x and y. Then since K has index one in $\underset{\sim}{F}(r, n, k)$, $\underset{\sim}{F}(r, n, k) = \langle x, y \rangle$. Notice that we can always replace x^{b_β} by $x^{b_{\bar\beta}}$ where $\bar\beta \equiv \beta \bmod n$ and $0 \leq \bar\beta < n$ since $x^{b_n} = 1$.

In addition to the relation $x^{b_n} = 1$ the modified Todd-Coxeter algorithm gives the following relations for K. From the subgroup generator $x = a_1a_2 \ldots a_{n-1}a_n$

$$x = \prod_{i=1}^{n} x^{b_{h(i-1)} - t}y, \tag{A}$$

and from the relation (m) of $\underset{\sim}{F}(r, n, k)$

$$\left\{\prod_{i=1}^{n} x^{b_{h(m-2+i)}-t}y\right\}^t x^{b_{h(m-1)} - b_{h(m-1)+1}} = 1. \tag{B_m}$$

Notice that (B_1) is the t-th power of relation (A). Now (B_m) and (B_{m+1}) together imply

$$yx^{b_{hm+1} - b_{hm}}y^{-1} = x^{b_{h(m-1)+1} - b_{h(m-1)}}.$$

But $b_{h(m-1)+1} - b_{h(m-1)} = n\,t\,b_{h(m-1)} + t$, and thus $y^{-1}x^{n\,t\,b_{h(m-1)} + t}y = x^{n\,t\,b_{hm} + t}$. Therefore

$$y^{-1}x^{t\left(\sum\limits_{s=1}^{h(m-1)}\binom{h(m-1)}{s}(nt)^s + 1\right)}y = x^{t\left(\sum\limits_{s=1}^{hm}\binom{hm}{s}(nt)^s + 1\right)}$$

that is,

$$y^{-1}x^{t(1+nt)^{h(m-1)}}y = x^{t(1+nt)^{hm}},$$

which may be written as

$$y^{-1}x^{tr^{h(m-1)}}y = x^{tr^{hm}}. \qquad (C_m)$$

However (C_m), $1 \leq m \leq n$ together with (A) and $x^{b_n} = 1$ imply (B_m), $1 \leq m \leq n$.

The relation (C_1) is $y^{-1}x^ty = x^{tr^h}$, and raising this relation to the power $r^{h(m-1)}$ gives the relation (C_m). Hence a presentation for K, and therefore for $\underset{\sim}{F}(r, n, k)$, is given by the generators x and y subject to the relations (A), (C_1) and $x^{b_n} = 1$. Since $b_{hi} - t$, $1 \leq i \leq n$, is divisible by t, relation (A) simplifies using (C_1) to give $y^n = x^\alpha$ where

$$\alpha = 1 - \sum_{i=1}^{n}(b_{hi} - t)r^{h(n-i)}.$$

$$\text{But} \quad b_{hi} - t = \left(\sum_{s=1}^{hi}\binom{hi}{s} n^s t^s - nt\right)/n,$$

$$= ((1 + nt)^{hi} - 1 - nt)/n,$$

$$= (r^{hi} - r)/n,$$

and so

$$\alpha = 1 - \sum_{i=1}^{n}((r^{hi} - r)r^{h(n-i)})/n,$$

$$= 1 - r^{hn} + \frac{r}{n}\sum_{i=1}^{n} r^{h(n-i)},$$

$$= \frac{r}{n}\sum_{i=1}^{n} r^{h(n-i)}, \quad \text{since } x^{r^{hn}} = x \text{ as } x^{(r^n - 1)/n} = 1.$$

Now since $x^{r^n} = x$, $r^{h(n-i)}$ may be written as r^{ν}, $0 \leq \nu < n-1$. Further since h and n are coprime, $r^{h(n-i)} = r^{h(n-j)}$ if, and only if, $i = j$. Therefore $\sum_{i=1}^{n} r^{h(n-i)} = \sum_{i=1}^{n} r^{(n-i)}$ and so

$$\alpha = \frac{r}{n} \sum_{i=1}^{n} r^{(n-i)},$$

$$= \frac{r(r^n - 1)}{n(r - 1)}.$$

Relation (A) becomes $y^n = x^{(r^n-1)/n(r-1)}$ since $r = 1 + (r - 1)$ and $x^{(r-1)(r^n-1)/(n(r-1))} = 1$. Relation (C_1) now simplifies using the modified relation (A). For $\alpha = vt + 1$ for some $v \in \mathbb{Z}$ since $b_{hi} - t$ is divisible by t, and so $y^{-1}x^{-vt}y = x^{-vtr^h}$ giving

$$y^{-1}xy = y^{-1}x^{-vt}yx^{\alpha} = x^{-vtr^h+\alpha} = x^{r^h} x^{\alpha(1-r^h)}.$$

Notice that we have used the fact that $x^{\alpha} \in Z(K)$, the centre of K, and so x commutes with y. However $x^{\alpha(1-r^h)} = 1$ since

$$\alpha(1 - r^h) = \frac{r(r^n - 1)(1 - r^h)}{n(r - 1)},$$

$$= \frac{(r - 1 + 1)(r^n - 1)(1 - r^h)}{n(r - 1)},$$

$$= \frac{(r^n - 1)(1 - r^h)}{n(r - 1)},$$

$$= \frac{u(r^n - 1)}{n} \qquad \text{for some } u \in \mathbb{Z}.$$

Thus $y^{-1}xy = x^{r^h}$.

Therefore the presentation for K is

$$\langle x,y \,|\, y^{-1}xy = x^{r^h},\ y^n = x^{(r^n-1)/(n(r-1))},\ x^{(r^n-1)/n} = 1 \rangle$$

as required.

Corollary 1. If $r \equiv 1 \bmod n$ then $\underset{\sim}{F}(r, n)$ is a metacyclic group of order precisely $r^n - 1$.

Proof. Take $k = 1$. Then from the first part of Theorem 4.5.1 the result is immediate.

Thus if $r \equiv 1 \bmod n$ we have in the first part of Theorem 4.5.1 solved the second part of problem 3 of (31) and extended the result to generalised Fibonacci groups.

Corollary 2. $\underset{\sim}{F}(r, n) \cong \langle x,y \mid y^{-1}xy = x^r,\ y^n = x^{(r^n-1)/(n(r-1))} \rangle$ where $r \equiv 1 \bmod n$.

Proof. Take $k = 1$, and hence $h = 1$, in the presentation for K. Then

$$K \cong \langle x,y \mid y^{-1}xy = x^r,\ y^n = x^{(r^n-1)/(n(r-1))},\ x^{(r^n-1)/n} = 1 \rangle .$$

We require to show that

$$y^{-1}xy = x^r, \tag{1}$$

and

$$y^n = x^{(r^n-1)/(n(r-1))}, \tag{2}$$

together imply

$$x^{(r^n-1)/n} = 1. \tag{3}$$

Raising (1) to the power $(r^n-1)/(n(r-1))$ gives

$$y^{-1}x^{(r^n-1)/(n(r-1))}y = x^{r(r^n-1)/(n(r-1))}.$$

But $x^{(r^n-1)/(n(r-1))} = y^n$ and so $x^{(r^n-1)/n(r-1))}$ is an element of $Z(K)$. Therefore

$$x^{(r^n-1)/(n(r-1))} = x^{r(r^n-1)/(n(r-1))},$$

that is

$$x^{(r^n-1)/n} = 1,$$

and so (3) is a consequence of (1) and (2).

Therefore we have now solved the first part of problem 3 of (31) so that the complete problem is solved by Corollary 1 and Corollary 2. If problem (3) of (31) is extended to a consideration of generalised Fibonacci groups then part of the problem is solved by Corollary 1 and the solution is completed in Theorem 4.5.2. First, however, we require another corollary to Theorem 4.5.1.

Corollary 3. $\underset{\sim}{F}(r, n) \cong \underset{\sim}{F}(r, n, k)$ when $r \equiv 1 \bmod n$ and $(k, n) = 1$.

Proof. Let Π be the set of prime factors of h.c.f. $(h, (r-1)(r^n-1))$ and λ the maximal Π'-number dividing $(r-1)(r^n-1)$, then $h + \lambda n$ is coprime to $(r-1)(r^n-1)$ and hence coprime to the order of x and the order of y. The group $\underset{\sim}{F}(r,n)$ has a presentation

$$\langle x,y \mid y^{-1}xy = x^r,\ y^n = x^{(r^n-1)/n(r-1))},\ x^{(r^n-1)/n} = 1 \rangle .$$

With this choice of λ, $x^{h+\lambda n}$ and $y^{h+\lambda n}$ together generate $\underset{\sim}{F}(r, n)$. Let $a = x^{h+\lambda n}$ and $b = y^{h+\lambda n}$. Then

$$\begin{aligned} b^{-1}ab &= y^{-(h+\lambda n)}\ x^{(h+\lambda n)}\ y^{(h+\lambda n)}, \\ &= y^{-h}x^{(h+\lambda n)}y^{h}, \qquad \text{since } y^n \text{ is a power of } x, \\ &= x^{(h+\lambda n)r^h}, \\ &= a^{r^h}. \end{aligned}$$

Also,

$$\begin{aligned} b^n &= y^{(h+\lambda n)n} \\ &= x^{(h+\lambda n)(r^n-1)/n(r-1))}, \\ &= a^{(r^n-1)/(n(r-1))}, \end{aligned}$$

and

$$\begin{aligned} a^{(r^n-1)/n} &= x^{(h+\lambda n)(r^n-1)/n}, \\ &= 1. \end{aligned}$$

Therefore $\underset{\sim}{F}(r, n)$ is a homomorphic image of $\underset{\sim}{F}(r, n, k)$. But from Theorem 4.5.1 we know that $|\underset{\sim}{F}(r, n)| = |\underset{\sim}{F}(r, n, k)|$ and therefore the result follows.

Theorem 4.5.2. Suppose $r \equiv 1 \bmod n$ and $(k, n) = 1$, then $\underset{\sim}{F}(r, n, k)$ has a 2-generator 2-relation presentation

$$\underset{\sim}{F}(r, n, k) \cong \langle x,y \mid y^{-1}xy = x^r, \quad y^n = x^{(r^n-1)/(n(r-1))} \rangle .$$

Proof. The result is an immediate consequence of Corollary 2 and Corollary 3 of Theorem 4.5.1.

4.6 Metacyclic Fibonacci-type Groups

In (15) a proof is given that if $r \equiv s \bmod n$ and $(r, n) = 1$, then $\underset{\sim}{H}(r, n, s)$ has the following 2-generator 2-relation presentation

$$\langle x,y \mid y^{-1}x^s y = x^r, \quad x^s = \prod_{i=1}^{n} y\, x^{(n-i-1)(r-s)/n} \rangle .$$

It is also shown in (15) that if, in addition, $(r, s) \neq 1$, then $\underset{\sim}{H}(r, n, s)$ is infinite and if $(r, s) = 1$ $\underset{\sim}{H}(r, n, s)$ is metacyclic of order $r^n - s^n$, being an extension of a cyclic group of order $(r^n - s^n)/n$ by a cyclic group of order n. Since $r \equiv s \bmod n$ we may assume that $r = nt + \alpha$, $s = nu + \alpha$. The method of proof is to let $x = a_1 a_2 \ldots a_n$, $y = (a_{\alpha+1}a_{\alpha+2} \ldots a_\alpha)^u(a_{\alpha+1}a_{\alpha+2} \ldots a_{2\alpha})$ and let $K = \langle x,y \rangle$. Then the modified Todd-Coxeter algorithm is used to show that K has index 1 in $\underset{\sim}{H}(r, n, s)$ and hence a presentation for $\underset{\sim}{H}(r, n, s)$ on the generators x and y may be obtained. In this section we generalise these results by showing that $\underset{\sim}{H}(r, n, k, s)$ is metacyclic if (i) $r \equiv s \bmod n$, (ii) $(r,n) = 1$, (iii) $(r + k - 1, n) = 1$ and a 2-generator 2-relation presentation is found for these groups. If, in addition, (iv) $(r,s) = 1$, then we show that $\underset{\sim}{H}(r, n, k, s)$ is a finite metacyclic group of order $r^n - s^n$. Note that when condition (iv) is added, condition (ii) becomes redundant.

Again the method of proof is to use the modified Todd-Coxeter algorithm but this time we show that $x = a_1a_2 \ldots a_n$ and $y = a_1a_2 \ldots a_{[\alpha+k-1]}$ generate $\underset{\sim}{H}(r, n, k, s)$. The proof given in (15) for the groups $\underset{\sim}{H}(r, n, s)$ does not generalise to the groups $\underset{\sim}{H}(r, n, k, s)$. However by taking $k = 1$ the proofs given in this section for the groups $\underset{\sim}{H}(r, n, k, s)$ do of course provide proofs for the groups $\underset{\sim}{H}(r, n, s)$.

In the next theorem we let $r = nt + \alpha$, $s = nu + \alpha$ as before. We also require the following notation. We shall use '[ ]' to denote that the expression in brackets is to be reduced modulo n so that [i] lies in the range $0 \leq i \leq n-1$. Also we shall take $\varepsilon_i = 1$ if $n \in \{i, i+1, \ldots, i+\alpha-1\}$, $1 \leq i \leq n$, where the elements of the set are reduced modulo n ($\varepsilon_i = 0$ otherwise) and shall define p_1 by $p_1 = \varepsilon_1 + \varepsilon_{1+\alpha} + \ldots + \varepsilon_k$.

Theorem 4.6.1. Suppose that

(i) $r \equiv s \bmod n$,

(ii) $(r, n) = 1$,

(iii) $(r + k - 1, n) = 1$,

then $\underset{\sim}{H}(r, n, k, s)$ has the 2-generator 2-relation presentation

$$\langle x,y \mid y^{-1}x^s y = x^r, \qquad 1 = \prod_{i=0}^{n-1} x^{[(n-i)\lambda]t - [(n-i-1)\lambda]u + p_{n-i}}\, y \rangle,$$

where $\lambda = [[r]^{-1}(k-1) + 1]$ and

$$p_i = p_1 \text{ if the integral part of } \frac{i\lambda}{n} \text{ is equal to the integral part of } \frac{(i-1)\lambda}{n} \text{ ~~for i = 1,2,...,n-1~~},$$
$$= p_1 - [r] \text{ otherwise.}$$

Proof. The relations of $\underset{\sim}{H}(r, n, k, s)$ are

$$(a_ja_{j+1} \ldots a_{j+n-1})^t(a_ja_{j+1} \ldots a_{j+\alpha-1})$$
$$((a_{j+\alpha+k-1}a_{j+\alpha+k} \ldots a_{j+\alpha+k-2})^u(a_{j+\alpha+k-1}a_{j+\alpha+k} \ldots a_{j+2\alpha+k-2}))^{-1} = 1,$$

$1 \leq j \leq n.$

Let $K = \langle x, y \rangle$ where $x = a_1 a_2 \ldots a_n$ and $y = a_1 a_2 \ldots a_{[\alpha+k-1]}$. We use the modified Todd-Coxeter coset enumeration algorithm to show that K has index 1 in $\underset{\sim}{H}(r, n, k, s)$ and hence we can obtain a presentation for $\underset{\sim}{H}(r, n, k, s)$ on the two generators x and y. We shall show that in fact we obtain a 2-generator 2-relation presentation. Define n cosets of K as follows: let coset $1 = K$, and for $2 \leq i \leq n$ define coset i by $i = (i - 1).a_{i-1}$. Then, since $x \in K$, $n.a_n = x.1$. From the subgroup generator y, we deduce the collapse $[\alpha + k] = y.1$, noting that $\alpha + k \neq 1$ since $(\alpha + k - 1, n) = 1$.

Since $(\alpha, n) = 1$, given i such that $1 \leq i \leq n$ there exists a unique j, $0 \leq j \leq n-1$ with $i \equiv j\alpha + 1 \bmod n$. For the rest of this proof we shall assume that $j\alpha + 1$ has been reduced modulo n. All integers representing cosets are reduced modulo n and all integers j appearing in expressions of the form $w(j)$ are reduced modulo n.

From the relations $i\alpha + 1$, $0 \leq i \leq n-3$ we deduce the collapses

$$(i + 2)\alpha + k = w((i + 1)\alpha + 1).(i + 1)\alpha + 1,$$

(where the w's are some words in the subgroup generators x and y)
and, since $(\alpha + k - 1, n) = 1$, we therefore have complete collapse. Thus $K = \langle x,y \rangle$ has index 1 in $\underset{\sim}{H}(r, n, k, s)$.

From the $((n-1)\alpha + 1)$st relation

$$((n + 1)\alpha + k) = w(n\alpha + 1)(n\alpha + 1),$$

that is $(\alpha + k) = w(n\alpha + 1).1$.

But $(\alpha + k) = y.1$ and so $w(n\alpha + 1) = y$.

We now write the jth relation, $1 \leq j \leq n$, in the form

$$(a_j a_{j+1} \ldots a_{j-1})^t (a_j a_{j+1} \ldots a_{j+\alpha-1})$$

$$= (a_{j+\alpha+k-1} a_{j+\alpha+k} \ldots a_{j+\alpha+k-2})^u (a_{j+\alpha+k-1} \ldots a_{j+2\alpha+k-2}),$$

$$= \rho(j) \quad \text{say.}$$

Consider the $(i\alpha + 1)$st relation. From the left hand side of the equality we obtain from the modified Todd-Coxeter algorithm

$$(i\alpha + 1).\rho(i\alpha + 1) = x^t x^{\varepsilon_{1+i\alpha}}.(i + 1)\alpha + 1, \tag{1}$$

and from the right hand side we obtain

$$\begin{aligned}(i\alpha + 1).\rho(i\alpha + 1) &= (w(i\alpha + 1))^{-1}((i + 1)\alpha + k).\rho(i\alpha + 1), \\ &= (w(i\alpha + 1))^{-1} x^u x^{\delta_{1+i\alpha}}.(i + 2)\alpha + k, \end{aligned} \tag{2}$$

where $\varepsilon_i = \begin{cases} 1 & \text{if } n \in \{i, i+1, \ldots, i+\alpha-1\}, \\ 0 & \text{otherwise,} \end{cases}$

and $\delta_i = \begin{cases} 1 & \text{if } n \in \{i+\alpha+k-1, i+\alpha+k, \ldots, i+2\alpha+k-2\}, \\ 0 & \text{otherwise.} \end{cases}$

Note that, by definition, $\delta_i = \varepsilon_{(i+\alpha+k-1)}$, where the subscripts are reduced modulo n.

From (1) and (2),

$$w(i\alpha + 1)x^t x^{\varepsilon_{1+i\alpha}}.(i + 1)\alpha + 1 = x^u x^{\delta_{1+i\alpha}}.(i + 2)\alpha + k.$$

Using this recurrence relation and the fact that $w(n\alpha + 1) = w(1) = y$, we obtain from the $((n - 1)\alpha + 1)$st relation

$$y = x^{-nu} x^{-\sum_{i=0}^{n-1} \delta_{1+i\alpha}} y\, x^{nt} x^{\sum_{i=0}^{n-1} \varepsilon_{1+i\alpha}}.$$

Since $(\alpha, n) = 1$, $\sum_{i=0}^{n-1} \delta_{1+i\alpha} = \sum_{i=0}^{n-1} \varepsilon_{1+i\alpha} = \alpha$.

Therefore $y = x^{-(nu+\alpha)} yx^{(nt+\alpha)}$, that is

$$y^{-1}x^s y = x^r. \tag{3}$$

In addition to relation (3) the modified Todd-Coxeter algorithm gives us just one further relation for K. The subgroup generator gave us the collapse $\alpha+k = 1$, and together with the n-2 relations $(i\alpha + 1), 0 \leq i \leq n-1$ gave us complete collapse. We have used the $((n-1)\alpha + 1)$st relation to obtain (3). Thus we require to use the $((n-2)\alpha + 1)$st relation in some form to find a second relation for K. We have then determined two relations for K. The results of section 2.3 then show that this is in fact a 2-generator 2-relation presentation for K. ~~From the ((n-2)α + 1)st relation~~ Now $\alpha + k = \omega(1).1$, $2\alpha+k = \omega(1+\alpha).\alpha+1$, $3\alpha+k = \omega(1+2\alpha).2\alpha+1$, and, since $(\alpha+k-1, n) = 1$,

$$\cancel{k = w((n-1)\alpha + 1).(n-1)\alpha + 1.}$$

$$1 = \omega((n-1)(\alpha+k-1)+1).(n-1)(\alpha+k-1)+1,$$

~~But we require to find an expression for (n − 1)α + 1. Since (i + 2)α + k = w((i + 1)α + 1).(i + 1)α + 1. let~~

(for $(n-1)(\alpha+k-1)+1 + (\alpha+k-1) \equiv 1 \bmod n$). Therefore

$$\cancel{(i + 2)\alpha + k = (n-1)\alpha + 1.}$$

$$1 = \omega((n-1)(\alpha+k-1)+1)\ \omega((n-2)(\alpha+k-1)+1).((n-2)(\alpha+k-1)+1).$$

~~Then $i = (n-2) - [\alpha^{-1}(\alpha + k - 1)]$.~~

~~Thus, $(n-1)\alpha + 1 = w((n-1)\alpha + 1 - [\alpha^{-1}(\alpha + k - 1)]).(n-1)\alpha+1-[\alpha^{-1}(\alpha+k-1)]$.~~

~~Let $\lambda = [[r]^{-1}(\alpha + k - 1)] = [\alpha^{-1}(\alpha + k - 1)]$ since $r = nt + \alpha$. Then~~

$$\cancel{k = w((n-1)\alpha + 1)\ w((n-1)\alpha + 1 - \lambda).(n-1)\alpha + 1 - \lambda.}$$

~~Following through the collapse and noting that since (α + k − 1, n) = 1 we get complete collapse, the second relation is thus~~

Proceeding thus we obtain

$$1 = \prod_{i=1}^{n} w((n-i)(\alpha + k - 1) + 1).$$

Now

$$w(i(\alpha+k-1) + 1) = x^{-(\delta_1+\delta_{1+\alpha}+\dots+\delta_{1+(i(1+\alpha^{-1}(k-1))-1)\alpha})} x^{-[i\lambda]u} y \; x^{[i\lambda t]} x^{\varepsilon_1 + \varepsilon_{1+\alpha}+\dots+\varepsilon_{1+(i(1+\alpha^{-1}(k-1)-1))\alpha}}.$$

Therefore

$$1 = x^{-[\lambda(n-1)]u+p_n} y \left(\prod_{i=1}^{n-2} x^{[\lambda(n-i)]t - [\lambda(n-i-1)]u + p_{n-i}} y\right) x^{[\lambda]t+p_1} y = 1,$$

where $p_1 = \varepsilon_1 + \varepsilon_{1+\alpha} + \ldots + \varepsilon_k$.

Now $p_n = -\delta_1 - \delta_{1+\alpha} - \delta_{1+2\alpha} - \ldots - \delta_{1+((n-1)(1+\alpha^{-1}(k-1)-1)\alpha}$. But $\delta_i = \varepsilon_{i+\alpha+k-1}$. Therefore $\delta_1 = \varepsilon_{\alpha+k}$, $\delta_{1+\alpha} = \varepsilon_{2\alpha+k}$, ..., $\delta_{2-2\alpha-k} = \varepsilon_{1-\alpha}$, and so $p_n - p_1 = -\sum_{i=0}^{n-1} \varepsilon_{1+i\alpha} = -\alpha$. Now p_i comes from looking at $w(i(\alpha+k-1)+1)$ and $w((i-1)(\alpha+k-1)+1)$. Therefore $p_i = \begin{cases} p_1, \\ p_1 - \alpha, \end{cases}$

being equal to p_1 if the integral part of $(i\lambda)/n$ is equal to the integral part of $(i-1)\lambda/n$ and to $p_1 - \alpha$ otherwise. Since $[n\lambda]t = 0$, we have

$$1 = \prod_{i=0}^{n-1} x^{[(n-i)\lambda]t - [(n-i-1)\lambda]u + p_{n-i}} y,$$

and hence $\underset{\sim}{H}(r, n, k, s)$ has a 2-generator 2-relation presentation

$$\langle x,y \mid y^{-1}x^s y = x^r,\ 1 = \prod_{i=0}^{n-1} x^{[(n-i)\lambda]t - [(n-i-1)\lambda]u + p_{n-i}} y \rangle.$$

<u>Corollary 1.</u> $H(r, n, s) \cong \langle x,y \mid y^{-1}x^s y = x^r,\ x^r = \prod_{i=0}^{n-1} x^{(n-i)t-(n-i-1)u} y \rangle$, where $r \equiv s \bmod n$ and $(r,n) = 1$.

<u>Proof.</u> From Theorem 4.6.1,

$$\underset{\sim}{H}(r, n, k, s) \cong \langle x,y \mid y^{-1}x^s y = x^r,\ 1 = \prod_{i=0}^{n-1} x^{[(n-i)\lambda]t-[(n-i-1)\lambda]u+p_{n-i}} y \rangle.$$

Let $k = 1$. Then $\lambda = 1$, $p_1 = \varepsilon_1 = 0$, $p_i = 0$, $1 \le i \le n-1$. Also $p_n = p_1 - \alpha = -\alpha$. In addition $[n]t = 0$. So

$$1 = x^{-(n-1)u-\alpha} y \prod_{i=1}^{n-1} x^{(n-i)t-(n-i-1)u} y.$$

But $r = nt + \alpha$. Therefore

$$x^r = \prod_{i=0}^{n-1} x^{(n-i)t - (n-i-1)u} y.$$

In the next corollary we show how we may now obtain the 2-generator 2-relation presentation for $\underset{\sim}{H}(r, n, s)$ obtained in (15).

Corollary 2. An alternative presentation for $\underset{\sim}{H}(r, n, s)$ is given by

$$\underset{\sim}{H}(r, n, s) \cong \langle a,b \mid b^{-1}a^s b = a^r,\ a^s = \prod_{i=1}^{n} ba^{(n-i-1)(r-s)/n} \rangle .$$

where $r \equiv s \bmod n$ and $(r, n) = 1$.

Proof. Let $b = yx^t$ and $a = x$ in Corollary 1. Then since $y^{-1}x^s y = x^r$,

$$y^{-1}x^s y = x^t x^r x^{-t},$$

that is

$$x^{-t}y^{-1}x^s y\, x^t = x^r,$$

or

$$b^{-1}a^s b = a^r.$$

Also $x^r = \prod_{i=0}^{n-1} x^{(n-i)t - (n-i-1)u}y$, that is

$$x^s = x^{s-r}x^{nt-(n-1)u} \left(\prod_{i=1}^{n-2} yx^t x^{(n-i-1)(t-u)}\right) yx^t y,$$

$$= \left(\prod_{i=1}^{n-2} yx^t x^{(n-i-1)(t-u)}\right) yx^t yx^t x^{u-t}.$$

But with $b = yx^t$ and $a = x$,

$$a^s = \left(\prod_{i=1}^{n-2} ba^{(n-i-1)(r-s)/n}\right) b^2\ a^{(s-r)/n},$$

$$= \prod_{i=1}^{n} ba^{\ (n-i-1)(r-s)/n}.$$

<u>Theorem 4.6.2.</u> If $r \equiv s \bmod n$ and $(r, n) = 1$, then if $(r, s) \neq 1$, $\underset{\sim}{H}(r, n, s)$ is infinite.

<u>Proof.</u> By Corollary 2 of Theorem 4.6.1,

$$\underset{\sim}{H}(r, n, s) \cong \langle a, b \mid b^{-1}a^{s}b = a^{r}, \quad a^{s} = \prod_{i=1}^{n} b\,a^{(n-i-1)(r-s)/n} \rangle .$$

Let $d = (r, s)$ and consider the homomorphic image L of $H(r, n, s)$ obtained by adding the relation $a^{d} = 1$. Then

$$L \cong \langle a, b \mid a^{d} = 1,\ b^{n} = 1 \rangle$$

and so is infinite.

The next result is used in the proof of Theorem 4.6.4.

<u>Lemma 4.6.3.</u> Let $(r, s) = 1$. Then if the group G has the presentation

$$G \cong \langle x, y \mid y^{-1}x^{s}y = x^{r}, \quad x^{r} = \prod_{i=0}^{n-1} x^{(n-i)t - (n-i-1)u}\, y \rangle$$

it follows that $y^{-1}xy = x^{\omega r}$ and $y^{n} = x^{\eta}$ where $(1 - \omega r)\eta \equiv \left(\frac{s-r}{n}\right)\left(\frac{1-(\omega r)^{n}}{1-\omega r}\right)$ mod $r^{n} - s^{n}$.

<u>Proof.</u> The relation $y^{-1}x^{s}y = x^{r}$ gives for any ξ, ζ, ~~ℓ, m~~ $\in \mathbb{Z}$; $\ell, m \geqslant 0$,

$$x^{\xi s^{\ell}}y^{\ell} = y^{\ell}x^{\xi r^{\ell}} \quad \text{and} \quad y^{m}x^{\zeta r^{m}} = x^{\zeta s^{m}}y^{m}.$$

Combining these

$$y^{m}x^{\zeta r^{m} + \xi s^{\ell}}y^{\ell} = x^{\zeta s^{m}}y^{m+\ell}x^{\xi r^{\ell}} .$$

But r^{m} is coprime to s^{ℓ} since $(r, s) = 1$, so, given any $\gamma \in \mathbb{Z} \setminus \{0\}$ we can find $\xi, \zeta \in \mathbb{Z}$ such that $\zeta r^{m} + \xi s^{\ell} = \gamma$.

This formula allows the y's to be collected in the second relation to give a relation of the form $y^{n} = x^{\eta}$ for some $\eta \in \mathbb{Z}$. Hence y^{n} is central in G. Since $y^{-n}x^{s^{n}}y^{n} = x^{r^{n}}$, we obtain $x^{r^{n}-s^{n}} = 1$. Let us denote $r^{n} - s^{n}$ by σ.

Then, $(s,\sigma) = 1$ since $(r, s) = 1$. Hence there exist $\omega, \tilde{\omega}$ such that $\omega s - \tilde{\omega}\sigma = 1$. But then

$$y^{-1}x^{\omega s}y = x^{\omega r},$$

so

$$y^{-1}x^{1+\tilde{\omega}\sigma}y = x^{\omega r},$$

that is $y^{-1}xy = x^{\omega r}$. In addition, of course, $x^{\sigma} = 1$.

The second relation is

$$x^r = \prod_{i=0}^{n-1} x^{(n-i)t - (n-i-1)u}\, y.$$

Using the relation $y^{-1}xy = x^{\omega r}$ we obtain, on writing $(n-i)t - (n-i-1)u = u + (n-i)(t-u)$, $y^n = x^{\eta}$ where

$$\eta = r - u - \sum_{i=1}^{n-1} u\,(\omega r)^i - \sum_{i=1}^{n-1} (t - u)i(\omega r)^i - n(t - u),$$

$$= r - \sum_{i=0}^{n-1} u(\omega r)^i - (t - u) \sum_{i=1}^{n-1} i\,(\omega r)^i - n(t - u),$$

$$= r - u\left(\frac{1 - (\omega r)^n}{1 - \omega r}\right) - \frac{(t - u)\omega r}{(1 - \omega r)^2} \{1 - n(\omega r)^{n-1} + (n-1)(\omega r)^n\} - n(t - u).$$

Therefore,

$$(1 - \omega r)\eta = (1 - \omega r)r - u(1 - (\omega r)^n) + \frac{(u-t)}{(1-\omega r)} (1 - (\omega r)^n)\omega r$$
$$+ n\,\frac{(u - t)}{(1 - \omega r)}\,(\omega r)^n\,(\omega r - 1) - n(t - u)(1 - \omega r).$$

But $(\omega r)^n \equiv 1 \bmod \sigma$, so modulo σ,

$$(1 - \omega r)\eta = (1 - \omega r)r + \frac{s-r}{n}\,\omega r\,\frac{(1 - (\omega r)^n)}{(1 - \omega r)} + (s-r)(-(\omega r)^n + 1 - \omega r),$$

$$= r - \omega r^2 + \omega r^2 - \omega s r + \frac{s-r}{n}\,\omega r\,\frac{(1 - (\omega r)^n)}{(1 - \omega r)}.$$

Since, in addition, $\omega s \equiv 1 \bmod \sigma$,

$$(1 - \omega r)\eta = \frac{s-r}{n} \omega r \left(\frac{1 - (\omega r)^n}{1 - \omega r}\right),$$

$$= -(1 - (\omega r)^n) \frac{(1 - \omega r)}{(1 - \omega r)}\left(\frac{(s - r)}{n}\right) + \frac{s - r}{n} \left(\frac{1 - (\omega r)^n}{1 - \omega r}\right),$$

$$= \left(\frac{s-r}{n}\right) \left(\frac{1 - (\omega r)^n}{1 - \omega r}\right).$$

Theorem 4.6.4. Suppose that

(i) $r \equiv s \bmod n$,

(ii) $(r + k - 1, n) = 1$,

(iii) $(r, s) = 1$,

then $\underset{\sim}{H}(r, n, k, s)$ is a finite metacyclic group of order $r^n - s^n$.

Proof. We show first that $(r, s) = 1$ implies that $(r, n) = 1$. For suppose the contrary and let $(r, n) = d$, $d \neq 1$. Since $r \equiv s \bmod n$, d divides s and so $(r, s) \neq 1$. Since $(r, n) = 1$ the three conditions of Theorem 4.6.1 are satisfied and so $\underset{\sim}{H}(r, n, k, s)$ has the 2-generator 2-relation presentation

$$\langle x,y \mid y^{-1}x^s y = x^r, \quad 1 = \prod_{i=0}^{n-1} x^{[(n-i)\lambda]t - [(n-i-1)\lambda]u + p_{n-i}} y \rangle .$$

Since $(r, s) = 1$ and as in the proof of Lemma 4.6.3 the relation $y^{-1}x^s y = x^r$ allows the y's to be collected in the second relation to give a relation of the form $y^n = x^\nu$. Hence y^n is central in $\underset{\sim}{H}(r, n, k, s)$ and since $y^{-n}x^{s^n}y^n = x^{r^n}$ we obtain $x^{r^n - s^n} = 1$. As before let us denote $r^n - s^n$ by σ. Then $y^{-1}xy = x^{\omega r}$ and $x^\sigma = 1$.

We wish to prove that the second relation which we have already shown may be reduced to the form $y^n = x^\nu$ is such that $\nu = \lambda\eta - \beta$ where $\beta(1 - \omega r)$ is divisible by σ, η is as in Lemma 4.6.3 and $\lambda = [\alpha^{-1}(\alpha + k - 1)]$.

For then the order of x

$$= ((1 - \omega r)\lambda\eta - (1 - \omega r)\beta, \sigma),$$
$$= ((1 - \omega r)\lambda\eta, \sigma),$$

since $(1 - \omega r)\beta$ is divisible by σ. Now λ is coprime to n since both α^{-1} and $\alpha + k - 1$ are coprime to n. Therefore the order of $x = ((1 - \omega r)\lambda\eta, \sigma) = ((1 - \omega r)\eta, \sigma)$. But by Lemma 4.6.3,

$$(1 - \omega r)\eta \equiv \left(\frac{s - r}{n}\right)\left(\frac{1 - (\omega r)^n}{1 - \omega r}\right) \mod \sigma.$$

Hence the order of x is $\left(\sigma, \left(\frac{r-s}{n}\right)\left(\frac{1 - (\omega r)^n}{1 - \omega r}\right)\right)$. The proof of the theorem is immediate when we have proved that the order of x is σ/n and that $\beta(1 - \omega r)$ is divisible by σ. Let us show first that the order of x is σ/n. Now

$$\frac{(r-s)}{n}\left(\frac{1 - (\omega r)^n}{1 - \omega r}\right) = \frac{1}{n}\left(r + \omega r^2 + \dots + \omega^{n-1}r^n - s - \omega rs - \dots - (\omega r)^{n-1}s\right).$$

Substituting $\omega s = 1 + \tilde{\omega}\sigma$ in the right hand side gives

$$\frac{(r-s)}{n}\left(\frac{1 - (\omega r)^n}{1 - \omega r}\right) = \frac{1}{n}\left(\omega^{n-1}r^n - s\right) - \tilde{\omega}r\,\frac{\sigma}{n}\left(1 + (\omega r) + \dots + (\omega r)^{n-2}\right).$$

But $r = s + jn$ for some $j \in \mathbb{Z}$; so we obtain

$$\left[\frac{1}{n}\left(\omega^{n-1}r^n - s\right) - \frac{\tilde{\omega}r\sigma}{n}\left(1 + (\omega s) + \dots + (\omega s)^{n-2}\right)\right] \mod \sigma$$

$$= \frac{1}{n}\left(\omega^{n-1}r^n - s\right) - \frac{\tilde{\omega}r\sigma}{n}\left(\frac{1 - (\omega s)^{n-1}}{1 - \omega s}\right).$$

Replacing ωs by $1 + \tilde{\omega}\sigma$ in the above expression, it reduces to

$$\left[\frac{1}{n}\left(\omega^{n-1}r^n - s\right) - \frac{\tilde{\omega}r\sigma}{n}(n-1)\right] \mod \sigma$$

$$\equiv \left[\frac{1}{n}\left(\omega^{n-1}r^n - s\right) + \frac{\tilde{\omega}r\sigma}{n}\right] \mod \sigma.$$

However s is coprime to σ and so the h.c.f. remains unchanged on multiplying through by s^{n-1}. We obtain

$$\frac{\sigma}{n} + \frac{r^n}{n}\left((1+\tilde{\omega}\sigma)^{n-1} - 1\right) + \frac{\tilde{\omega}rs^{n-1}}{n}\sigma = \left(\frac{\sigma}{n} + \frac{r^n}{n}(n-1)\tilde{\omega}\sigma + \frac{\tilde{\omega}r^n}{n}\sigma\right) \bmod \sigma$$

$$= \frac{\sigma}{n} \bmod \sigma.$$

Therefore, since $\left(\sigma, \frac{(r-s)}{n}\left(\frac{1-(\omega r)^n}{1-\omega r}\right)\right) = \left(\sigma, \frac{\sigma}{n}\right)$ the order of x is, as required, $\frac{\sigma}{n}$.

To complete the theorem it remains to show that $\beta(1-\omega r)$ is divisible by σ. Consider the second relation in the presentation for $\underset{\sim}{H}(r, n, k, s)$, that is

$$1 = \prod_{i=0}^{n-1} x^{[(n-i)\lambda]t - [(n-i-1)\lambda]u + p_{n-i}}\, y.$$

We proceed as follows. Let $i\lambda = \gamma_i n + [i\lambda]$. Then

$$\begin{aligned}
&[i\lambda]t - [(i-1)\lambda]u + p_i \\
&\quad = i\lambda t - \gamma_i nt - (i-1)\lambda u + \gamma_{i-1}nu + p_i, \\
&\quad = i\lambda t - \gamma_i nt - (i-1)\lambda u + \gamma_{i-1}nu - \gamma_i\alpha + \gamma_{i-1}\alpha + p_1, \\
&\quad = (i\lambda t - (i-1)\lambda u) - \gamma_i(nt+\alpha) + \gamma_{i-1}(nu+\alpha) + p_1, \\
&\quad = (i\lambda t - (i-1)\lambda u) - \gamma_i r + \gamma_{i-1}s + p_1, \qquad 1 \le i \le n-1.
\end{aligned}$$

Now $\gamma_1 = 0$ since λ is reduced mod n. Put $\gamma_o = 0$ for convenience. Then

$$1 = x^{-[(n-1)\lambda]u + p_1 - \alpha}\, y \prod_{i=1}^{n-1} x^{(n-i)\lambda t - (n-i-1)\lambda u - \gamma_{n-i}r + \gamma_{n-i-1}s + p_1}\, y.$$

The general term in the product on the right hand side may be written as $x^{(n-i)\lambda t - (n-i-1)\lambda u}\, x^{-\gamma_{n-i}r + \gamma_{n-i-1}s + p_1}y$. If we write $-[(n-1)\lambda]u + p_1 - \alpha$ $= -(n-1)\lambda u - \lambda\alpha + \gamma_{n-1}nu + (\lambda-1)\alpha + p_1 = -\lambda r + n\lambda t - (n-1)\lambda u + \gamma_{n-1}nu + (\lambda-1)\alpha + p_1$ and if we take each of the n-1 expressions $x^{-\gamma_{n-i}r + \gamma_{n-i-1}s + p_1}$ to the right hand side of the relation by means of the relation $y^{-1}xy = x^{\omega r}$ then we obtain

$$x^{\lambda r} = \left(\prod_{i=0}^{n-1} x^{(n-i)\lambda t - (n-i-1)\lambda u}y\right) x^{\beta},$$

where $\beta = \sum_{i=1}^{n-1} (-\gamma_i r + \gamma_{i-1} s + p_1)(\omega r)^i + \gamma_{n-1}\, nu + (\lambda - 1)\alpha + p_1.$

But $\gamma_{i-1}\omega s r \equiv \gamma_{i-1} r \bmod \sigma$ so we get

$$\beta = -\gamma_{n-1}\, r(\omega r)^{n-1} + \gamma_{n-1}\, nu + (\lambda - 1)\alpha + p_1 \sum_{i=0}^{n-1} (\omega r)^i.$$

Since $(\omega r)^n \equiv 1 \bmod \sigma$,

$$\beta = (\omega r)^{n-1}(-\gamma_{n-1} r + \gamma_{n-1}\, nu\, \omega r + (\lambda - 1)\alpha\omega r) + p_1 \sum_{i=0}^{n-1} (\omega r)^i.$$

However $nu = s - \alpha$, so

$$\beta = (\omega r)^{n-1}(-\alpha\, \gamma_{n-1}\omega r + (\lambda - 1)\alpha\omega r) + p_1 \sum_{i=0}^{n-1} (\omega r)^i.$$

But $\gamma_{n-1} = \lambda - 1$ since $(n - 1)\lambda = (\lambda - 1)n + n - \lambda$ and $n\, \gamma_{n-1} = (n - 1)\lambda - [(n - 1)\lambda]$. Therefore

$$\begin{aligned} \beta &= p_1 \sum_{i=0}^{n-1} (\omega r)^i, \\ &= p_1 \left(\frac{1 - (\omega r)^n}{1 - \omega r}\right), \end{aligned}$$

and so $(1 - \omega r)\beta = p_1(1 - (\omega r)^n)$.

Also,

$$\begin{aligned} (p_1(1 - (\omega r)^n), \sigma) &= (p_1 s^n(1 - (\omega r)^n), \sigma) \text{ since } (s^n, \sigma) = 1, \\ &= (p_1(s^n - r^n), \sigma), \\ &= \sigma. \end{aligned}$$

Thus the second relation is

$$x^{\lambda r} = \left\{\prod_{i=0}^{n-1} x^{(n-i)\lambda t - (n-i-1)\lambda u}y\right\} x^{\beta}, \tag{†}$$

where $(1 - \omega r)\beta$ is divisible by σ. But by Lemma 4.6.3

$$x^{r} = \prod_{i=0}^{n-1} x^{(n-i)t - (n-i-1)u}y$$

may be written as

$$y^{n} = x^{\eta}.$$

Therefore

$$x^{\lambda r} = \prod_{i=0}^{n-1} x^{(n-i)\lambda t - (n-i-1)\lambda u}y$$

may be written as

$$y^{n} = x^{\lambda\eta}.$$

Relation (†) then becomes

$$x^{\lambda\eta - \beta} = y^{n}$$

which is the required result. ~~This~~ Thus with the conditions of the theorem as stated

$$\underset{\sim}{H}(r, n, k, s) \cong \langle x,y \mid y^{-1}x^{s}y = x^{r},\ \ x^{(r^{n}-s^{n})/n} = 1,\ \ y^{n} = x^{\lambda\eta-\beta}\rangle$$

and $\underset{\sim}{H}(r, n, k, s)$ is metacyclic of order $r^{n} - s^{n}$, being an extension of a cyclic group of order $(r^{n} - s^{n})/n$ by a cyclic group of order n.

Next we show how the results of Theorem 4.6.4 can be extended in the special case of the groups $\underset{\sim}{F}(r, n, k)$ to the class of groups $\underset{\sim}{F}(r, n, k, h)$ so that given certain conditions $F(r, n, k, h)$ is metacyclic of order $r^n - 1$. We also consider whether there are similar results for the class of groups $\underset{\sim}{H}(r, n, k, s, h)$, $s \neq 1$.

<u>Theorem 4.6.5.</u> Suppose that $(r - 1)h \equiv 0 \bmod n$ and $(k, n) = 1$, then

$$\underset{\sim}{F}(r, n, k, h) \cong \underset{\sim}{F}(r^{(n,h)}, d, k\gamma)$$

where $d = n/(n,h)$ and γ is the unique multiplicative inverse of $h/(n,h)$ mod d.

<u>Proof.</u> By Lemma 4.2.2 we can assume without loss of generality that $k = 1$. The first relation of $\underset{\sim}{F}(r, n, 1, h)$ reduces to

$$(a_h\ a_{2h} \cdots a_{dh})^{(r-1)/d} a_h = a_{h+1},$$

where the generators $a_h, a_{2h}, \ldots, a_{dh}$ are all distinct. This allows us to express a_{h+1} in terms of $a_h, a_{2h}, \ldots, a_{dh}$ and relation $(1 + ih)$ allows us to express $a_{(i+1)h+1}$ also in terms of $a_h, a_{2h}, \ldots, a_{dh}$ for $1 \leq i \leq d-1$. Substituting these expressions in relation (2) gives

$$(a_h\ a_{2h} \cdots a_{dh})^{(r^2-1)/d} a_h = a_{h+2}.$$

Continuing in this way be obtain

$$(a_h\ a_{2h} \cdots a_{dh})^{(r^j-1)/d} a_h = a_{h+j}, \qquad 1 \leq j \leq (n,h),$$

since a_{h+j}, $1 \leq j \leq (n,h)$ are all distinct and $a_{h+(n,h)} \in \{a_h, a_{2h}, \ldots, a_{dh}\}$. At this stage the n relations for $\underset{\sim}{F}(r, n, 1, h)$ have been reduced to the d relations

$$((a_h\ a_{2h} \cdots a_{dh})^{(r^{(n,h)}-1)/d} a_h\ a_{h+(n,h)}^{-1})\theta^{(i-1)h} = 1, \qquad 1 \leq i \leq d.$$

Putting $x_i = a_{ih}$, $1 \leq i \leq d$ we obtain the relations

$$((x_1 x_2 \ldots x_d)^{(r^{(n,h)}-1)/d} x_1 x_{1+\gamma}^{-1})\bar{\theta}^{i-1} = 1, \qquad 1 \leq i \leq d$$

where $\bar{\theta}$ permutes the subscripts of x_i, $1 \leq i \leq d$, according to the permutation $(1\ 2\ \ldots\ d)$. Now $(n, h) = \beta n + \gamma h$ and so $(n, h) \equiv \gamma h \bmod n$. Therefore $1 \equiv (\gamma h/(n,h)) \bmod d$ as required.

Corollary 1. With the conditions on r, n, k, h as in the statement of Theorem 4.6.5, $\underset{\sim}{F}(r, n, k, h)$ is metacyclic of order r^n-1.

Proof. The result follows from Theorem 4.6.4 and Theorem 4.6.5 on showing that $r^{(n,h)} \equiv 1 \bmod d$ and γ is coprime to n. Now $\gamma h/(n,h) \equiv 1 \bmod d$. Therefore $(h/(n,h), n/(n,h)) = 1$. Also $r^{(n,h)}-1 = (r-1)(1+r+r^2+ \ldots +r^{(n,h)-1})$ and so $r^{(n,h)}-1 \equiv 0 \bmod d$ if $(r-1) \equiv 0 \bmod d$. Now $(r-1)h \equiv 0 \bmod n$ and therefore $(r-1)h/(n,h) = \lambda n/(n,h)$. Thus, since $d = n/(n,h)$, $(r-1)h/(n,h) \equiv 0 \bmod d$. Since we have proved that $(h/(n,h), d) = 1$ it follows that $(r-1) \equiv 0 \bmod d$ and so $r^{(n,h)} \equiv 1 \bmod d$.

We ~~new~~ must show that $(k\gamma, d) = 1$. Let $n = (n,h)\alpha$ and suppose $(\gamma, n) = \tilde{k} \neq 1$. Then $\tilde{k}$ divides α and $\tilde{k}$ divides γ. That is $\tilde{k}$ divides γ and $\tilde{k}$ divides $n/(n,h)$. Now $(n,h) = \beta n + \gamma h$ and so $\tilde{k}$ divides (n,h). Therefore $\tilde{k}^2$ divides n and $\tilde{k}$ divides h. From this we can now show that $\tilde{k}^2$ divides (n,h) which implies that $\tilde{k}^3$ divides n and $\tilde{k}^2$ divides h. Proceeding in this way we find that all powers of $\tilde{k}$ divide (n,h). Therefore $\tilde{k} = 1$ and $(\gamma, n) = 1$. But $(k,n)=1$ so the result follows

An obvious question is to ask whether Theorem 4.6.5 and Corollary 1 of that theorem can be extended to the class of groups $\underset{\sim}{H}(r, n, k, s, h)$, $s \neq 1$. In fact Theorem 4.6.5 may be generalised and we obtain the following result.

Theorem 4.6.6. $\underset{\sim}{H}(r, n, k, s, h) \cong \underset{\sim}{H}\left(\frac{r^{(n,h)}}{s^{(n,h)-1}}, d, k\gamma, s\right)$ if r is divisible by s, $(r-s)h \equiv 0 \bmod n$ and k is coprime to n, where $d = n/(n, h)$ and $k\gamma$ is such that $h(k\gamma+s-1)/(n,h) \equiv 1 \bmod d$.

Note that a necessary condition for this theorem is that r is divisible by s, whereas in the statement of Theorem 4.6.5 $(r,s) = 1$. We therefore omit the proof of Theorem 4.6.6 since it does not allow us to extend the result of Corollary 1 of Theorem 4.6.5.

4.7 On a conjecture.

In this section we consider the groups $\underset{\sim}{H}(r, 4, 2)$. We first show that if r is even $\underset{\sim}{H}(r, 4, 2)$ is infinite. In (15) E.F. Robertson and I conjecture that the groups $\underset{\sim}{H}(r, 4, 2)$, r odd are metacyclic. We show how a 2-generator 2-relation presentation may be obtained for $\underset{\sim}{H}(r, 4, 2)$ r odd. We had checked the conjecture for $r \leq 23$ by using the coset enumeration programme (1) which also enabled us to obtain for r = 3,5,7,9 and 11, the orders of $\underset{\sim}{H}(r, 4, 2)$, these being 5,15,125,1015 and 4095 respectively. The truth of the conjecture has since been proved by A.M. Brunner and the proof will appear in (6). We conclude the section by stating a version of his result.

<u>Lemma 4.7.1.</u> If r is even, $\underset{\sim}{H}(r, 4, 2)$ is infinite.

<u>Proof.</u> This is an immediate consequence of ~~Lemma 4.2.11~~ Theorem 4.2.12.

We now consider $\underset{\sim}{H}(r, 4, 2)$, r odd, giving first a presentation for $\underset{\sim}{H}(r, 4, 2)$.

<u>Theorem 4.7.2.</u> If r is odd $\underset{\sim}{H}(r, 4, 2)$ has a presentation

$$\langle x,y \mid (xy)^{(r-1)/2} = y^2x^{-1}yx^{-1},\ (yx)^{(r-1)/2} = x^2y^{-1}xy^{-1} \rangle .$$

<u>Proof.</u> We consider first the case when $r = 4n + 1$.

$$\underset{\sim}{H}(r, 4, 2) \cong \langle a,b,c,d \mid (abcd)^n a = bc,\ (bcda)^n b = cd,$$
$$(cdab)^n c = da,\ (dabc)^n d = ab \rangle .$$

Let $x = ab$ and $y = cd$ and let $K = \langle x,y \rangle$. We use the modified Todd-Coxeter coset enumeration algorithm to show that K has index 1 in $\underset{\sim}{H}(r, 4, 2)$, that is x and y generate $\underset{\sim}{H}(r, 4, 2)$. We are thus able to find a presentation for $\underset{\sim}{H}(r, 4, 2)$ on the generators x and y.

Let coset 1 = K and define cosets 2 and 3 as follows: 2 = 1.a and 3 = 1.c. Then since x and y $\in$ K, 2b = x.1 and 3d = y.1. From the second relation, considering coset 2, we deduce the collapse $2 = (xy)^n xy^{-1}.1$ and from the fourth relation, considering coset 3, we deduce the collapse $3 = (yx)^n yx^{-1}.1$. Therefore we have complete collapse giving

$$1.a = (xy)^n xy^{-1}.1, \qquad 1.b = yx^{-1}(xy)^{-n}x.1,$$
$$1.c = (yx)^n yx^{-1}.1, \qquad 1.d = xy^{-1}(yx)^{-n}y.1.$$

From the first and third relations of the original presentation we obtain, using the modified algorithm, see 2.2, the relations for K,

$$(xy)^n(xy)^n xy^{-1} = yx^{-1}(xy)^{-n}x(yx)^n yx^{-1},$$

and

$$(yx)^n(yx)^n yx^{-1} = xy^{-1}(yx)^{-n}y(xy)^n xy^{-1},$$

that is

$$K \cong \langle x,y \mid (xy)^{2n} = y^2x^{-1}yx^{-1},\ (yx)^{2n} = x^2y^{-1}xy^{-1} \rangle.$$

or

$$K \cong \langle x,y \mid (xy)^{(r-1)/2} = y^2x^{-1}yx^{-1},\quad (yx)^{(r-1)/2} = x^2y^{-1}xy^{-1} \rangle.$$

Now we consider the case when r = 4n + 3.

$$\underset{\sim}{H}(r,\ 4,\ 2) \cong \langle a,b,c,d \mid (abcd)^n abc = da,\ (bcda)^n bcd = ab,$$
$$(cdab)^n cda = bc,\ (dabc)^n dab = cd \rangle.$$

As in the first case let x = ab and y = cd and let K = $\langle x,y \rangle$. Let coset 1 = K and define cosets 2 and 3 as follows: 2 = 1.a and 3 = 1.c. Then since x and y $\in$ K, 2.b = x.1 and 3.d = y.1. From the second relation, considering coset 2, we deduce in this case the collapse $2 = (xy)^n xyx^{-1}.1$ and from the fourth relation, considering coset 3, we deduce the collapse $3 = (yx)^n yxy^{-1}.1$. Again we have complete collapse giving

$$1.a = (xy)^n\, xy\, x^{-1}.1, \qquad 1.b = xy^{-1}x^{-1}(xy)^{-n}x.1,$$

$$1.c = (yx)^n y\, xy^{-1}.1, \qquad 1.d = yx^{-1}y^{-1}(yx)^{-n}y.1.$$

From the first and third relations of the original presentation we obtain the relations for K,

$$(xy)^n x(yx)^n yxy^{-1} = yx^{-1}y^{-1}(yx)^{-n}y(xy)^n xyx^{-1},$$

and

$$(yx)^n y(xy)^n xyx^{-1} = xy^{-1}x^{-1}(xy)^{-n}x(yx)^n yxy^{-1},$$

that is

$$K \cong \langle x,y \mid (xy)^{2n+1} = y^2x^{-1}yx^{-1},\ (yx)^{2n+1} = x^2y^{-1}xy^{-1} \rangle,$$

or

$$K \cong \langle x,y \mid (xy)^{(r-1)/2} = y^2x^{-1}yx^{-1},\ (yx)^{(r-1)/2} = x^2y^{-1}xy^{-1} \rangle.$$

Thus, when r is odd, $\underset{\sim}{H}(r, 4, 2)$ has a presentation

$$\langle x,y \mid (xy)^{(r-1)/2} = y^2x^{-1}yx^{-1},\ (yx)^{(r-1)/2} = x^2y^{-1}xy^{-1} \rangle.$$

Lemma 4.7.3. If r is odd, $\underset{\sim}{H}(r, 4, 2)$ has a presentation

$$\langle t,y \mid (ty^2)^{(r-1)/2} = yt^{-2},\ y^2t^{-1} = (tyt)^2 \rangle.$$

Proof. From Theorem 4.7.2

$$\underset{\sim}{H}(r, 4, 2) \cong \langle x,y \mid (xy)^{(r-1)/2} = y^2x^{-1}yx^{-1},\ (yx)^{(r-1)/2} = x^2y^{-1}xy^{-1} \rangle.$$

Now let $xy^{-1} = t$. Then since $\underset{\sim}{H}(r, 4, 2)$ is generated by x and y, so it is generated by y and t.

The first relation becomes

$$(ty^2)^{(r-1)/2} = y^2y^{-1}t^{-1}yy^{-1}t^{-1} = yt^{-2},$$

and the second relation becomes

$$(yty)^{(r-1)/2} = tyt^2.$$

Thus,

$$\underset{\sim}{H}(r, 4, 2) \cong \langle t,y \,|\, (ty^2)^{(r-1)/2} = yt^{-2}, (yty)^{(r-1)/2} = tyt^2 \rangle,$$

or, simplifying,

$$\underset{\sim}{H}(r, 4, 2) \cong \langle t,y \,|\, (ty^2)^{(r-1)/2} = yt^{-2},\ y^2t^{-1} = (tyt)^2 \rangle.$$

Before giving the result of Brunner we give a lemma concerning $\underset{\sim}{H}(r, 4, 2)/\underset{\sim}{H}'(r, 4, 2)$, r odd.

Lemma 4.7.4. If r is odd, $\underset{\sim}{H}(r, 4, 2)/\underset{\sim}{H}'(r, 4, 2)$ is isomorphic to $\mathbb{Z}_{5(r-2)}$ if r-2 is not divisible by 5 and is isomorphic to $\mathbb{Z}_5 \times \mathbb{Z}_{r-2}$ if r-2 is divisible by 5.

Proof. By Lemma 4.7.3 $\underset{\sim}{H}(r, 4, 2)$, r odd, has a presentation

$$\langle a,b \,|\, (ab^2)^{(r-1)/2} = ba^{-2},\ b^2a^{-1} = (aba)^2 \rangle.$$

Therefore

$$\underset{\sim}{H}(r, 4, 2)/\underset{\sim}{H}'(r, 4, 2) \cong \langle a,b \,|\, a^{(5+r-2)/2}\, b^{r-2} = 1,\ a^5 = 1, [a,b]=1 \rangle.$$

Since r is odd, then if r-2 is not divisible by 5, (5 + r - 2)/2 and r-2 are coprime. Therefore

$$\underset{\sim}{H}(r, 4, 2)/\underset{\sim}{H}'(r, 4, 2) \cong \mathbb{Z}_{5(r-2)}.$$

If r-2 is divisible by 5 then $(5 + r - 2)/2$ is divisible by 5, and so

$$\underset{\sim}{H}(r, 4, 2)/\underset{\sim}{H}'(r, 4, 2) \cong \mathbb{Z}_5 \times \mathbb{Z}_{r-2}.$$

The final theorem which shows that the groups $\underset{\sim}{H}(r, 4, 2)$, r odd, are metacyclic is due to A.M. Brunner. It was given first in a private communication and will appear in (6). Note that the presentation for $\underset{\sim}{H}(r, 4, 2)$ required is that obtained in Lemma 4.7.3. The result may be stated as follows:

<u>Theorem 4.7.5.</u> Let $G \cong \langle a,b \mid (ab^2)^{(r-1)/2} = ba^{-2},\ b^2a^{-1} = (aba)^2 \rangle$ where r is odd; then ~~it may be shown that~~

$$G \cong \langle a,b \mid b^{r-2} = a^{-4k/5},\ \ b^{-1}ab = a^{(-1)^{(r-3)/2}2^{r-2}-1},\ a^k = 1 \rangle,$$

where $k = 2^{r-2} + (-1)^{(r-1)(r-3)/8}2^{(r-1)/2} + 1$.

Appendix 1

The programme CCRG

```
C THE PROGRAMME CCRG
C RESULTS SUMMARY RECORD
      IMPLICIT INTEGER*2(A-Z)
      DIMENSION COSET(128,10),CINV(128,10)
      DIMENSION START(128,10),STINV(128,10)
      DIMENSION END(128,10),ENDIN(128,10)
      DIMENSION RELTLE(1000),WORD(4000),WINV(4000)
      DIMENSION ANSWER(4000),ABANS(32)
      READ(5,311) RBKT
      WRITE(6,312) RBKT
  311 FORMAT(41)
  312 FORMAT(I10)
      READ(5,10) NCOSET,NGGEN,NGEN
   10 FORMAT(4X,I5,13X,I5,23X,I5,25X)
      WRITE(6,10)NCOSET,NGGEN,NGEN
C WORD RECORDS
      IP2=0
      DO 31 I=1,NCOSET
      DO 31 J=1,NGGEN
      READ(5,20) IW,JW,KW,NTERM
   20 FORMAT(I3,I3,I3,I3)
      WRITE(6,20)IW,JW,KW,NTERM
      IP1=IP2+1
      IP2=2*NTERM+IP1-1
      COSET(IW,JW)=KW
      CINV(KW,JW)=IW
      START(IW,JW)=IP1
      END(IW,JW)=IP2
      STINV(KW,JW)=IP1
      ENDIN(KW,JW)=IP2
      IF(NTERM.EQ.0) GO TO 31
      READ(5,30) (WORD(IP),IP=IP1,IP2)
   30 FORMAT(24I3)
      WRITE(6,30)(WORD(IP),IP=IP1,IP2)
C TO FIND INVERSE OF A WORD
      J1=(IP1+1)/2
      J2=IP2/2
      DO 32 P=J1,J2
```

```
      WINV(2*P-1)=WORD(2*J2-2*(P-J1)-1)
      WINV(2*P)=-WORD(2*J2-2*(P-J1))
   32 CONTINUE
      WRITE(6,150) (WINV(P),P=IP1,IP2)
   31 CONTINUE
  150 FORMAT(' WINV',20I4)
C READ RELATIONS
      READ(5,40) NGGEN,NGEN,NREL
   40 FORMAT(I4,4X,I4,I4)
      WRITE(6,40)NGGEN,NGEN,NREL
      DO 53 K=1,NREL
      READ(5,41) IRLTH
   41 FORMAT(I4)
      WRITE(6,41)IRLTH
      READ(5,50) (RELTLE(KK),KK=1,IRLTH)
   50 FORMAT(19 I4)
      WRITE(6,50)(RELTLE(KK),KK=1,IRLTH)
      WRITE(6,51)
   51 FORMAT(' RELATION')
C TO WORK OUT ANSWER FOR A RELATION
      DO 53 M=1,NCOSET
      LANS=0
      I=M
      I2=0
      DO 120 J=1,IRLTH
      IF(RELTLE(J).GT.0) GO TO 101
      IF(RELTLE(J).LT.0) GO TO 102
      IF(RELTLE(J).EQ.0) GO TO 103
  103 WRITE(6,104)
  104 FORMAT(' PROGRAM IS WRONG')
  101 IR=RELTLE(J)
      I1=LANS+1
      I2=LANS+1+END(I,IR)-START(I,IR)
      IF(I2.LT.I1) GO TO 107
      IS=START(I,IR)
      DO 105 II=I1,I2
      ISS=IS-1+II-LANS
      ANSWER(II)=WORD(ISS)
```

```
  105 CONTINUE
  107 I=COSET(I,IR)
      GO TO 120
  102 IR=-RELTLE(J)
      I1=LANS+1
      I2=LANS+1+ENDIN(I,IR)-STINV(I,IR)
      IF(I2.LT.I1) GO TO 108
      IS=STINV(I,IR)
      DO 106 II=I1,I2
      ISS=IS-1+II-LANS
      ANSWER(II)=WINV(ISS)
  106 CONTINUE
  108 I=CINV(I,IR)
  120 LANS=I2
      IF(I2.EQ.0) GO TO 110
  110 CONTINUE
C CONDENSE ANSWER
C FOR THE PROGRAMME CORGAB WE REPLACE THE
C SECTION CONDENSE ANSWER BY ABELIANISE ANSWER
      J=1
      IA=ANSWER(1)
      IC=ANSWER(2)
      DO 70 I=3,LANS,2
      J=J+2
      IB=ANSWER(I)
      IF(IB.NE.IA) GO TO 170
      IC=IC+ANSWER(I+1)
      IF(IC.EQ.0) GO TO 270
      ANSWER(J-1)=IC
      J=J-2
      GO TO 70
  270 J=J-4
      IF(J.LT.1) GO TO 500
      IA=ANSWER(J)
      IC=ANSWER(J+1)
      GO TO 70
  500 IA=-1
      GO TO 70
```

```
  170 ANSWER(J)=IB
      IC=ANSWER(I+1)
      ANSWER(J+1)=IC
      IA=IB
   70 CONTINUE
      LANS=J+1
C END OF CONDENSE ANSWER
C OUTPUT
      IF(LANS.LT.2) GO TO 553
      DO 80 I=1,LANS,2
      IF(ANSWER(I).EQ.0) ANSWER(I)=-16128
      IF(ANSWER(I).EQ.1) ANSWER(I)=-15872
      IF(ANSWER(I).EQ.2) ANSWER(I)=-15616
      IF(ANSWER(I).EQ.3) ANSWER(I)=-15360
      IF(ANSWER(I).EQ.4) ANSWER(I)=-15104
      IF(ANSWER(I).EQ.5) ANSWER(I)=-14848
      IF(ANSWER(I).EQ.6) ANSWER(I)=-14592
   80 CONTINUE
      WRITE(6,350)
  350 FORMAT(' CONDENSED ANSWER')
      LANS=LANS+1
      ANSWER(LANS)=RSKT
      WRITE(6,351) (ANSWER(I),I=1,LANS)
      WRITE(7,351) (ANSWER(I),I=1,LANS)
  351 FORMAT(' ('19(A1,I3)/('  '19(A1,I3)))
      WRITE(6,52)
   52 FORMAT(' END OF ANSWER')
   53 CONTINUE
      CALL EXIT
  553 WRITE(6,54)
   54 FORMAT(' WORD IS TRIVIAL')
      GO TO 53
      END
```

<u>The programme CCRGAB - the abelianised answer</u>

```
C ABELIANISED ANSWER
      DO 400 J=1,NGEN
      INDEX=0
      IF(LANS.EQ.0) GO TO 553
      DO 401 I=1,LANS,2
      IF(ANSWER(I).EQ.J-1) INDEX=INDEX+ANSWER(I+1)
  401 CONTINUE
      ABANS(2*J-1)=J-1
      ABANS(2*J)=INDEX
  400 CONTINUE
      I=2
      LANS=2*NGEN
  404 IF(I.GT.LANS) GO TO 405
      IF(ABANS(I).NE.0) GO TO 402
      I1=I+1
      DO 403 J=I1,LANS
  403 ABANS(J-2)=ABANS(J)
      LANS = LANS - 2
      GO TO 404
  402 I=I+2
      GO TO 404
  405 IF(LANS.EQ.0) GO TO 553
      DO 500 I=1,LANS
  500 ANSWER(I)=ABANS(I)
```

Appendix 2

Reference (8)

Some examples using coset enumeration

C. M. CAMPBELL

Introduction. A modification of the Todd–Coxeter coset enumeration process [1] has been described by Campbell [2], Moser [3], and Benson and Mendelsohn [4]. In this note we give some examples that illustrate the way in which this modification is used.

Let G be an abstract group with a finite number of generators and relations, and let H be a subgroup of G. Assume further that the index $[G : H]$ of H in G is finite. Let E denote the identity and let $(^-)$ denote the inverse of an element.

THEOREM. *If from the relation $R = E$, where E is the identity and*

$$R = a_1 \ldots a_r \ldots a_s \ldots a_p, \quad 1 \leqslant r \leqslant s \leqslant p,$$

we win the new information

$$\alpha . a_r a_{r+1} \ldots a_s = \beta,$$

where each a_i is a generator g_j or its inverse and α, β are integers denoting cosets, then

$$\alpha . a_r \ldots a_s = W . \beta,$$

where
$$W = W_{r-1} W_{r-2} \ldots W_1 W_p \ldots W_{s+1}$$

is a word in the subgroup and α, β are now thought of as coset representatives.

Proof. Express the relation $R = E$ in the form

$$a_r \ldots a_s = \bar{a}_{r-1} \bar{a}_{r-2} \ldots \bar{a}_1 \bar{a}_p \ldots \bar{a}_{s+1}.$$

Then

$$\alpha . a_r \ldots a_s = \alpha . \bar{a}_{r-1} \bar{a}_{r-2} \ldots \bar{a}_1 \bar{a}_p \ldots \bar{a}_{s+1}.$$

From previous information in the tables we find $\alpha \cdot \bar{a}_{r-1}$ expressed in the form $W_{r-1} \cdot \gamma$ (α and γ are now thought of as coset representatives and W_{r-1} is a word in the subgroup H):

$$\alpha . a_r \ldots a_s = W_{r-1} \gamma . \bar{a}_{r-2} \ldots \bar{a}_1 \bar{a}_p \ldots \bar{a}_{s+1}.$$

Now, again from the tables, $\gamma \cdot \bar{a}_{r-2} = W_{r-2} \cdot \delta$.

Therefore

$$\alpha . a_r \ldots a_s = W_{r-1} W_{r-2} \delta . \bar{a}_{r-3} \ldots \bar{a}_1 \bar{a}_p \ldots \bar{a}_{s+1}.$$

Repeating the process,

$$\alpha.a_r \dots a_s = W_{r-1}W_{r-2} \dots W_1W_p \dots W_s\mu.\bar{a}_{s+1}.$$

Finally, from the tables,

$$\mu.\bar{a}_{s+1} = W_{s+1}.\beta.$$

Therefore

$$\alpha.a_r \dots a_s = W_{r-1}W_{r-2} \dots W_1W_p \dots W_{s+1},$$

and hence

$$\alpha.a_r \dots a_s = W.\beta.$$

The Todd–Coxeter process leads to an enumeration table and from the modification and our theorem (with $p = 1$) we obtain a table carrying additional information (see Example 1).

Examples. In [2] an algorithmic proof is given to show that the two relations

$$RS^2 = S^3R, \quad SR^2 = R^3S$$

imply that $R = S = E$, where E is the identity. This has been generalized by Benson and Mendelsohn [4] who show that the two relations

$$RS^{n-1} = S^nR, \quad SR^{n-1} = R^nS$$

again imply that $R = S = E$.

We consider two examples that arise from the previous two.

EXAMPLE 1. Let $G = \{R, S, T, U\}$ be subject to the relations $RS = S^2T$, $ST = T^2U$, $TU = U^2R$, $UR = R^2S$. Then G is cyclic of order 5.

Proof.	$RS = S^2T$	(1)
	$ST = T^2U$	(2)
	$TU = U^2R$	(3)
	$UR = R^2S$	(4)
	$ST^2U = RS$	(5) from (1), (2)
	$STU = \bar{R}$	(6) from (3), (4), (5)

where, as before, $(^-)$ denotes the inverse

$SU = \bar{R}ST$	(7) from (3), (6)
$S^3T = E$	(8) from (1), (2), (6)
$SRS = E$	(9) from (1), (8)
$U^2 = R$	(10) from (4), (9)
$ST = \bar{R}^2U$	(11) from (6), (10)
$S = \bar{R}^3$	(12) from (7), (11)
$U = \bar{R}^2$	(13) from (4), (12)
$T = \bar{R}$	(14) from (11), (12), (13)
$R^5 = E$	(15) from (10)

This algebraic proof follows algorithmically from the modified Todd–Coxeter process with the following enumeration and information tables. New information is found from the underlined positions in the tables in the order numbered.

	R	S	$\bar{T}$	$\bar{S}$	$\bar{S}$
1	1	2(2)3		2	1
2	(7)	1		3	2
3					3
4	2	3			4
5					5

	S	T	$\bar{U}$	$\bar{T}$	$\bar{T}$
1	2	5			1
2	3	2	(3)	5	2
3	(6)	1	5	2	3
4					4
5					5

	T	U	$\bar{R}$	$\bar{U}$	$\bar{U}$
1				4	1
2	5	1	1	5(5)2	
3	2				3
4					4
5		2	4	1(4)5	

	U	R	$\bar{S}$	$\bar{R}$	$\bar{R}$
1	4(1)2		1	1	1
2	5				2
3					3
4(8)1		1		2	4
5	1	1			5

Coset	Representative	R	S	T	U
$1 = \{R\}$	$1 = E$	$1.R = R.1$	$1.S = E.2$		$1.U = E.4$
$2 = 1.S$	$2 = S$		$2.S = E.3$	$2.T = E.5$	$2.U = \bar{R}.5$
$3 = 2.S$	$3 = S^2$			$3.T = R.2$	
$4 = 1.U$	$4 = U$	$4.R = R^2.2$			$4.U = R.1$
$5 = 2.T$	$5 = ST$				$5.U = \bar{R}.1$

From the positions numbered 3, 6, 7 we have, using our theorem, the additional information

$$5.TU = R.2,$$
$$3.ST = E.1,$$
$$2.RS = E.1.$$

In the above tables $4.U = R.1$ and $5.U = \bar{R}.1$, which implies that 4 and 5 are the same coset, and in terms of coset representatives $5 = \bar{R}^2.4$. Replacing 5 by $\bar{R}^2.4$ in the information tables gives

R	S	T	U
$1.R = R.1$	$1.S = E.2$		$1.U = E.4$
	$2.S = E.3$	$2.T = \bar{R}^2.4$	$2.U = \bar{R}^3.4$
		$3.T = R.2$	
$4.R = R^2.2$			$4.U = R.1$

From these tables $1.U = E.4$ and $2.U = \bar{R}^3.4$, and it now follows that 1 and 2 are the same coset. Repeating the process as before leads finally to complete collapse.

The new information $4.R = 2$ and $3.T = 2$ reduce to equations (1) and (4) but from the new information $5.TU = 2$ equation (5) is obtained. In the first of the two calculations below we work with the coset representative as an integer and in the second we think of the coset representatives as a word in the group.

$5.TU = 5.\bar{T}ST$	$ST.TU = ST.\bar{T}ST$	from (2)
$= E2.ST$	$= ES.ST$	
$= EE3.T$	$= EES^2.T$	
$= EER.2$	$= EER.S$	
$5.TU = R.2$	$= R.S$	from (1)

This is equation (5). In a similar manner we obtain equations (6)–(10). Equation (11) comes from the first coincidence when cosets 4 and 5 are identified.

$5 = \bar{R}1.\bar{U}$	from	$5.U = R.1$
$= \bar{R}\bar{R}4.U\bar{U}$	from	$4.U = R.1$
$= \bar{R}^2.4,$		

or, in terms of coset representatives,

$ST = \bar{R}E.\bar{U}$	from (6)
$= \bar{R}\bar{R}UU\bar{U}$	from (10)
$= \bar{R}^2U,$	

and this is equation (11). From the other coincidences we obtain equations (12)–(14).

Example 2. The relations $SR^2 = RSRS$ and $RS^2 = SRSR$ imply that $R = S = E$.

Proof. The proof is again obtained algorithmically as in Example 1.

$SR^2 = RSRS$	(1)
$RS^2 = SRSR$	(2)
$SR^3 = R^2S^2$	(3) from (1), (2)
$S^2RSR = \bar{R}SR^2S$	(4) from (1), (2)
$S^3 = \bar{R}S^2R^2$	(5) from (1), (3)
$SR^2SRS = RS^2R$	(6) from (1), (3)
$S^2R^2S = S^2R^2$	(7) from (2), (3), (5), (6)

whence $S = E$ and $R = E$.

The following question now arises. Given $SR^n = R^{n-1}SRS$ and $RS^n = S^{n-1}RSR$, do these relations imply $R = S = E$? (True for $n = 1, 2$.) One further example is the following: show that the group generated by five generators a, b, c, d, e subject only to the relations $ab = c$, $bc = d$, $cd = e$, $de = a$, $ea = b$, is cyclic of order 11. This problem was discussed in the *American Mathematical Monthly* [5].

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THE ORDERS OF CERTAIN METACYCLIC GROUPS

C. M. CAMPBELL AND E. F. ROBERTSON

Reference (10)

1. *Introduction*

Let F_n be the free group on $\{a_i : i \in \mathbf{Z}_n\}$, where the set of congruence classes mod n is used as an index set for the generators. Let ϕ be the permutation $(1\,2\,3 \ldots n)$ of $\mathbf{Z}_n$ and denote by θ the automorphism of F_n induced by ϕ, namely

$$a_i \theta = a_{i\phi}.$$

Suppose w is a word in F_n. Let $N(w)$ be the normal closure of $\{w\theta^i : 1 \leqslant i \leqslant n\}$ in F_n. Define the group $G(w)$ by $G(w) = F_n/N(w)$ and call $w\theta^{i-1} = 1$ the relation (i) of $G(w)$.

When, for some integer $r \geqslant 2$, the word w is given by

$$w = a_1 a_2 a_3 \ldots a_r a_{r+1}^{-1}$$

then $G(w)$ is the *Fibonacci group* $\mathrm{F}(r, n)$. If r and k are integers with $r \geqslant 2$, $k \geqslant 0$ and w is the word

$$w = a_1 a_2 a_3 \ldots a_r a_{r+k}^{-1}$$

then $G(w)$ is the *generalised Fibonacci group* $\mathrm{F}(r, n, k)$. The group $\mathrm{F}(r, n)$ is the group $\mathrm{F}(r, n, 1)$. The groups $\mathrm{F}(r, n)$ were introduced by Conway in [5] and are studied in [4], [6], [8], [9], and [10]. When $r \equiv 1 \pmod n$, a bound for the order of $\mathrm{F}(r, n)$ is known (see [9]) but it has been an open question as to what is the exact order. In this paper we solve this problem by showing that the order of $\mathrm{F}(r, n)$ is $r^n - 1$ and thus the bound given in [9] is never attained. We also solve a problem arising from results on generalised Fibonacci groups presented in [4].

The main tools used in this investigation are the Todd-Coxeter coset enumeration algorithm, see for example [7], and the modification to the algorithm described in [2].

2. *The orders of the groups*

THEOREM. *Suppose* $r \equiv 1 \pmod n$ *and* k *is coprime to* n, *then* $\mathrm{F}(r, n, k)$ *is metacyclic of order* $r^n - 1$.

Proof. Let $x = a_1 a_2 a_3 \ldots a_n$, let $H = \langle x \rangle$ and use the modified Todd-Coxeter algorithm to find the index of H in $\mathrm{F}(r, n, k)$ and also a presentation for H. Using a similar notation to that described in [3], define cosets $2, 3, \ldots, n$ by $i.a_i = i+1$,

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$1 \leqslant i \leqslant n-1$. The subgroup generator gives us $n.a_n = x.1$. Since $r \equiv 1 \pmod{n}$, $r = nt+1$, say. Define $b_0 = 0, b_1 = t$ and inductively $b_j = (nt+1)\, b_{j-1}+t$.

Using relation (1) we obtain $1.a_{k+1} = x^{b_1}.2$, and, in general, from relation (r) we obtain

$$r.a_{k+r} = x^{b_1}.(r+1) \qquad 1 \leqslant r \leqslant n-1.$$

Also relation (n) gives $n.a_k = x^{b_1+1}.1$. We may now use relation $(k+1)$ to obtain the information $1.a_{2k+1} = x^{b_2}.2$. Proceeding successively in this way we obtain from relation $((\alpha-1)\, k+i)$ the information

$$i.a_{\alpha k+i} = \begin{cases} x^{b_\alpha}.(i+1), & 1 \leqslant i \leqslant n-1, \\ x^{b_\alpha+1}.1 & i = n, \end{cases}$$

and since k is coprime to n the coset enumeration terminates showing the index of H in $\mathrm{F}(r, n, k)$ to be n and H to be normal in $\mathrm{F}(r, n, k)$. The modified Todd-Coxeter algorithm gives a presentation for H, see [1], $H = \langle x : x^{b_n} = 1 \rangle$.

Assume by induction that

$$b_{j-1} = \sum_{s=1}^{j-1} \binom{j-1}{s} n^{s-1}\, t^s.$$

Then

$$b_j = (nt+1) \sum_{s=1}^{j-1} \binom{j-1}{s} n^{s-1}\, t^s + t$$

$$= \sum_{s=1}^{j} \binom{j}{s} n^{s-1}\, t^s.$$

Hence

$$b_n = \sum_{s=1}^{n} \binom{n}{s} n^{s-1}\, t^s$$

and so $|\mathrm{F}(r, n, k)| = n.b_n = r^n - 1$.

This theorem explains the isomorphism between F(6, 5, 1) and F(6, 5, 3) investigated in [4].

COROLLARY. *If* $r \equiv 1 \pmod{n}$ *then* $\mathrm{F}(r, n)$ *is a metacyclic group of order precisely* $r^n - 1$.

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Mathematical Institute,
University of St. Andrews,
St. Andrews, KY16 9SS.

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ON METACYCLIC FIBONACCI GROUPS

by C. M. CAMPBELL and E. F. ROBERTSON

Reference (11)

(Received 14th November 1973)

1. Introduction

Let F_n be the free group on $\{a_i: i \in \mathbf{Z}_n\}$, where the set of congruence classes mod n is used as an index set for the generators. Let ϕ be the permutation $(1, 2, 3, \ldots, n)$ of $\mathbf{Z}_n$ and denote by θ the automorphism of F_n induced by ϕ, namely

$$a_i\theta = a_{i\phi}.$$

Let r and k be integers such that $r \geqq 2$, $k \geqq 0$ and let N be the normal closure of the set

$$\{(a_1a_2 \ldots a_r a_{r+k}^{-1})\theta^m\colon 1 \leqq m \leqq n\}$$

in F_n. Define the *generalised Fibonacci group* $F(r, n, k)$ by

$$F(r, n, k) = F_n/N.$$

We shall call $(a_1a_2\ldots a_ra_{r+k}^{-1})\theta^{m-1} = 1$ the relation (m) of $F(r, n, k)$. The groups $F(r, n, 1)$ are the *Fibonacci groups* discussed in (3), where it is proved that these groups are metacyclic if $r \equiv 1 \bmod n$. In (3) two questions are posed relating to the case $r \equiv 1 \bmod n$, namely to find the orders of these groups and also 2-generator 2-relation presentations for them. The first of these questions was solved in (2) and in this paper we solve the second problem.

The generalised Fibonacci groups $F(r, n, k)$ are discussed in (1) where it is stated that $F(6, 5, 3)$ is isomorphic to $F(6, 5, 1)$. We prove a generalisation of this result, namely that when $r \equiv 1 \bmod n$ $F(r, n, k)$ is isomorphic to $F(r, n, 1)$ for any k coprime to n.

2. A 2-generator presentation for $F(r, n, k)$

Suppose $r \equiv 1 \bmod n$ and let $r = nt+1$. Define $b_0 = 0$, $b_1 = t$ and, inductively, $b_j = rb_{j-1}+t$. Denote by x the element $a_1a_2\ldots a_n$ of $F(r, n, k)$. In (2) we showed that $\langle x \rangle$ has index n in $F(r, n, k)$ when k is coprime to n and that x has order b_n. A coset enumeration was carried out using the modified Todd-Coxeter algorithm. Defining the cosets $2, 3, \ldots, n$ by $i.a_i = i+1$, $1 \leqq i \leqq n-1$, the following relations between coset representatives were obtained

$$i.a_{\alpha k+i} = \begin{cases} x^{b_\alpha}.(i+1) & 1 \leqq i \leqq n-1, \\ x^{b_\alpha+1}.1 & i = n. \end{cases}$$

Let $y = a_{1+k}$. We show that x and y together generate $F(r, n, k)$ and obtain the following theorem.

Theorem 1. *Let $r \equiv 1 \bmod n$ and let k be coprime to n. Then*

$$F(r, n, k) = \langle x, y \mid y^{-1}xy = x^{r^h}, y^n = x^{(r^n-1)/(n(r-1))}, x^{(r^n-1)/n} = 1\rangle,$$

where $hk \equiv 1 \bmod n$ and $1 \leqq h \leqq n-1$.

Proof. We use the modified Todd-Coxeter algorithm for the subgroup $H = \langle x, y\rangle$, making use of the results already stated for $\langle x\rangle$ and following through the collapses which occur with the addition of the information $1.a_{1+k} = y.1$. Using relation (1) we obtain $1.a_{1+k} = x^t.2$ and so $2 = x^{-t}y.1$. But $1.a_{\alpha k+1} = x^{b_\alpha}.2 = x^{b_\alpha - t}y.1$. Since k is coprime to n,

$$1.a_i = x^{b_{h(i-1)}-t}y.1; \quad 1 \leqq i \leqq n.$$

Therefore the subgroup H has index one in $F(r, n, k)$ and thus

$$F(r, n, k) = \langle x, y\rangle.$$

Notice that we can always replace x^{b_β} by $x^{b_{\bar\beta}}$ where $\bar\beta \equiv \beta \bmod n$ and $0 \leqq \bar\beta < n$ since $x^{b_n} = 1$.

In addition to the relation $x^{b_n} = 1$ the modified Todd-Coxeter algorithm gives us the following relations for H. From the subgroup generator $x = a_1a_2\ldots a_{n-1}a_n$ we obtain

$$(A) \qquad x = \prod_{i=1}^{n} x^{b_{h(i-1)}-t}y,$$

and from the relation (m) of $F(r, n, k)$ we obtain

$$(B_m) \qquad \left(\prod_{i=1}^{n} x^{b_{h(m-2+i)}-t}y\right)^t x^{b_{h(m-1)}-b_{h(m-1)+1}} = 1.$$

Notice that (B_1) is the tth power of relation (A). Now (B_m) and (B_{m+1}) together imply

$$yx^{b_{hm+1}-b_{hm}}y^{-1} = x^{b_{h(m-1)+1}-b_{h(m-1)}}.$$

But $b_{h(m-1)+1}-b_{h(m-1)} = ntb_{h(m-1)}+t$, and thus $y^{-1}x^{ntb_{h(m-1)}+t}y = x^{ntb_{hm}+t}$. We therefore obtain

$$(C_m) \qquad y^{-1}x^{tr^{h(m-1)}}y = x^{tr^{hm}}.$$

However (C_m), $1 \leqq m \leqq n$, together with (A) imply (B_m), $1 \leqq m \leqq n$.

The relation (C_1) is $y^{-1}x^ty = x^{tr^h}$, and raising this relation to the power $r^{h(m-1)}$ gives the relation (C_m). Hence a presentation for H, and therefore for $F(r, n, k)$, is given by the generators x and y subject to the relations (A), (C_1) and $x^{b_n} = 1$. Since $b_{hi}-t$, $1 \leqq i \leqq n$, is divisible by t, relation (A) simplifies using (C_1) to give $y^n = x^\alpha$ where

$$\alpha = 1 - \sum_{i=1}^{n} (b_{hi}-t)r^{h(n-i)}.$$

But $b_{hl}-t = (r^{hl}-r)/n$ and so $\alpha = r(r^n-1)/(n(r-1))$. Relation (A) becomes $y^n = x^{(r^n-1)/(n(r-1))}$ since $r = 1+(r-1)$ and $x^{(r-1)(r^n-1)/(n(r-1))} = 1$. Relation (C_1) now simplifies using the modified relation (A). For $\alpha = vt+1$ for some $v \in \mathbf{Z}$, and so $y^{-1}x^{-vt}y = x^{-vtr^h}$ giving

$$y^{-1}xy = x^{-vtr^h+\alpha} = x^{r^h}x^{\alpha(1-r^h)}.$$

Notice we have used the fact that $x^\alpha \in Z(H)$, the centre of H. However, $x^{\alpha(1-r^h)} = 1$ since

$$\alpha(1-r^h) = \frac{r^n-1}{n} \cdot \frac{(1-r^h)}{r-1} = u \cdot \frac{r^n-1}{n}$$

for some $u \in \mathbf{Z}$. Thus $y^{-1}xy = x^{r^h}$.

Corollary 1. *$F(r, n, 1) = \langle x, y \mid y^{-1}xy = x^r,\ y^n = x^{(r^n-1)/(n(r-1))}\rangle$, where $r \equiv$ mod n.*

Proof. It suffices to show that the relations $y^{-1}xy = x^r$ and

$$y^n = x^{(r^n-1)/(n(r-1))}$$

together imply $x^{(r^n-1)/n} = 1$. Raising $y^{-1}xy = x^r$ to the power $(r^n-1)/(n(r-1))$ gives

$$y^{-1}x^{(r^n-1)/(n(r-1))}y = x^{r(r^n-1)/(n(r-1))}.$$

Since $x^{(r^n-1)/(n(r-1))} \in Z(H)$, $x^{(r^n-1)/n} = 1$.

Corollary 2. *$F(r, n, 1) \cong F(r, n, k)$ when $r \equiv 1 \bmod n$ and k is coprime to n.*

Proof. Let Π be the set of prime factors of h.c.f. $(h, (r-1)(r^n-1))$ and λ the maximal Π'-number dividing $(r-1)(r^n-1)$, then $h+\lambda n$ is coprime to $(r-1)(r^n-1)$ and hence coprime to the order of x and the order of y. The group $F(r, n, 1)$ has a presentation

$$\langle x, y \mid y^{-1}xy = x^r,\ y^n = x^{(r^n-1)/(n(r-1))},\ x^{(r^n-1)/n} = 1\rangle.$$

With this choice of λ, $x^{h+\lambda n}$ and $y^{h+\lambda n}$ together generate $F(r, n, 1)$. With $a = x^{h+\lambda n}$, $b = y^{h+\lambda n}$ it is straightforward to check that

$$b^{-1}ab = a^{r^h}, \quad b^n = a^{(r^n-1)/(n(r-1))}, \quad a^{(r^n-1)/n} = 1.$$

Hence $F(r, n, 1)$ is a homomorphic image of $F(r, n, k)$ and, using the fact proved in (2) that $|F(r, n, 1)| = |F(r, n, k)|$, the result follows.

An immediate consequence of Corollary 1 and Corollary 2 is the following theorem.

Theorem 2. *Let $r \equiv 1 \bmod n$ and let k be coprime to n. Then $F(r, n, k)$ has a 2-generator 2-relation presentation*

$$F(r, n, k) = \langle x, y \mid y^{-1}xy = x^r,\ y^n = x^{(r^n-1)/(n(r-1))}\rangle.$$

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Mathematical Institute
University of St. Andrews
St Andrews KY16 9SS

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REMARKS ON A CLASS OF 2-GENERATOR GROUPS OF DEFICIENCY ZERO

C. M. CAMPBELL and E. F. ROBERTSON

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1. Introduction

Let G be a finitely presented group. A finite presentation $\mathscr{P}$ of G is said to have deficiency $m - n$ if it defines G with m generators and n relations. The deficiency of G is the maximum of the deficiencies of all the finite presentations $\mathscr{P}$ of G. If G is finite the deficiency of G is less than or equal to zero. The only finite two generator groups of deficiency zero that are known are certain metacyclic groups given by Wamsley (1970), a class of nilpotent groups given by Macdonald in (1962) and a class of groups given by Wamsley (1972).

In this paper we consider a class of two generator groups of deficiency zero. Define the group $\mathrm{G}(m, n)$, where m, n are non-zero integers, by

$$\mathrm{G}(m,n) = \langle a, b \,|\, [a^m, b^{-1}] = a^{-1}b^n a,\ [b^m, a^{-1}] = b^{-1}a^n b\rangle.$$

The groups $\mathrm{G}(1, n)$ are metacyclic groups of order n^3 and exponent n^2. We investigate the groups $\mathrm{G}(2, n)$ for $-3 \leqq n \leqq 5$ showing that $\mathrm{G}(2, -3)$ is a group of order $2^8.3^3$ and that $\mathrm{G}(2, 3)$ is a group of order $2^{15}.3^3$. These two groups are neither metabelian nor nilpotent nor are they isomorphic to any of the groups in the classes defined in Macdonald (1962), and Wamsley (1970), Wamsley (1972). We also show that $\mathrm{G}(3, 1)$ is isomorphic to $SL(2, 5)$. This answers a question posed in Campbell (1969) as to whether the groups $\mathrm{G}(m, 1)$ are all trivial.

The main tools used in this investigation are the Todd-Coxeter coset enumeration algorithm, see for example Coxeter and Moser (1972), and the modification to the algorithm described in Benson and Mendelsohn (1966). We would like to take this opportunity to thank Dr. M. J. Beetham for allowing us to use his coset enumeration programme Beetham (unpublished). The machine calculations were carried out on the IBM 360 computer of the University of St. Andrews.

2. The groups G (m,n)

We use the notation Z_n for the cyclic group of order n. It is easy to see that, if $\mathrm{G}'(m, n)$ is the derived group of $\mathrm{G}(m, n)$, then $\mathrm{G}(m, n)/\mathrm{G}'(m, n)$ is isomorphic to $\mathrm{Z}_n \times \mathrm{Z}_n$ if $n \geqq 1$ and to $\mathrm{Z}_{-n} \times \mathrm{Z}_{-n}$ if $n \leqq 1$.

Consider $G(1, n)$ for $n \geqq 1$. This group has the presentation

$$G = G(1,n) = \langle a, b \mid ab^{-1} = b^{n-1}a, ba^{-1} = a^{n-1}b \rangle.$$

Hence $a^{n-1}b.b^{n-1}a = 1$ giving $a^n = b^{-n}$. Thus $a^n \in Z(G)$, the centre of G. Since $G/\langle a^n \rangle \simeq Z_n \times Z_n$ we must have $\langle a^n \rangle = G'$. Now

$$bab^{-1} = b^n a = a^{1-n}.$$

Hence $b^n ab^{-n} = a^{1-n^2}$ and since $b^n ab^{-n} = a, a^{n^2} = 1$. Therefore G has order n^3 and is metacyclic of exponent n^2. If $n \leqq 1$ a similar argument holds. We have the following theorem.

THEOREM 1. *If* $n \geqq 1$, $G(1, n)$ *is a finite metacyclic group of order* n^3. *The centre of* $G(1, n)$ *is equal to the derived group of* $G(1, n)$ *and is cyclic of order* n. *The group* $G(1, -n)$ *is isomorphic to* $G(1, n)$.

Let $d = \text{h.c.f.}(m, n)$. It is easy to see that $G(m, n)$ has a homomorphic image isomorphic to $Z_d * Z_d$, the free product of two copies of Z_d. For, if $a^d = 1$, $b^d = 1$ are added to $G(m, n)$, the relations $[a^m, b^{-1}] = a^{-1}b^n a$ and $[b^m, a^{-1}] = b^{-1}a^n b$ are then redundant. In the case $m = 2$ we can prove a slightly stronger result.

THEOREM 2. $G(m, n)$ *has* $Z_d * Z_d$ *as a homomorphic image, where* $d = \text{h.c.f.}(m, n)$. *Therefore* $G(m, n)$ *is infinite if* m *and* n *are not coprime. In* $G(2, n)$, *for* n *even, the subgroup* $\langle a^2, b^2 \rangle$ *is normal.* $G(2, n)/\langle a^2, b^2 \rangle$ *is isomorphic to* D_∞, *the infinite dihedral group.*

PROOF. Since $D_\infty \simeq Z_2 * Z_2$ we need only prove that $\langle a^2, b^2 \rangle$ is normal in $G(2, n)$. Since n is even put $n = 2k$. It is sufficient to prove that $ab^2a^{-1} \in \langle a^2, b^2 \rangle$. But

$$\begin{aligned} b^2 &= a^{-1}ba^{2k}ba \\ &= a^{-1}(ba^2b^{-1})^k b^2 a \\ &= a^{-1}(ab^{2k}a)^k b^2 a \\ &= (b^{2k}a^2)^{k-1}\, b^{2k}(ab^2a^{-1})a^2. \end{aligned}$$

Hence $ab^2a^{-1} \in \langle a^2, b^2 \rangle$ as required.

Next we give a result about $G(m, n)$ where m, n are coprime.

THEOREM 3. *Let* m *and* n *be coprime. Then* $G(m, n)$ *has* $G(1, n)$ *as a homomorphic image. In particular* $G(-n, n + 1)$ *is isomorphic to* $G(1, n + 1)$.

PROOF. Consider the group

$$H = \langle a, b \mid [a^m, b^{-1}] = a^{-1}b^n a, [b^m, a^{-1}] = b^{-1}a^n b, a^n b^n = 1 \rangle.$$

Clearly H is a homomorphic image of $G(m, n)$.

Since $a^n = b^{-n}$, a^n and b^n are in $Z(H)$, the centre of H. Hence

$$b^{-m}ab^{m}a^{-1} = a^n \Rightarrow ab^{m}a^{-1} = b^{m}b^{-n}.$$

Thus $a^n b^m a^{-n} = b^m b^{-n^2}$ and so $b^{n^2} = 1$. But

$$\langle a, b \mid a^n = b^n = 1, [b^m, a^{-1}] = 1, [a^m, b^{-1}] = 1\rangle \simeq \mathbf{Z}_n \times \mathbf{Z}_n,$$

since m is coprime to n^2 and so $[b^m, a^{-1}] = 1$ implies that a and b commute. Therefore $H' = Z(H) = \langle a^n\rangle$ and H is isomorphic to $\mathrm{G}(1, n)$.

Now consider

$$\mathrm{G}(-n, n+1) = \langle a, b \mid [a^{-n}, b^{-1}] = a^{-1}b^{n+1}a, [b^{-n}, a^{-1}] = b^{-1}a^{n+1}b\rangle.$$

The relation $[a^{-n}, b^{-1}] = a^{-1}b^{n+1}a$ gives $a^{n+1}ba^{-n}b^{-1} = b^{n+1}a$. Also $b^{n+1}ab^{-n}a^{-1} = a^{n+1}b$ and so $a^{n+1}ba^{-n}b^{-(n+1)}\,a^{-1} = a^{n+1}b$ giving $a^{-(n+1)} = b^{n+1}$. Therefore by the first part of the theorem $\mathrm{G}(-n, n+1)$ is isomorphic to $\mathrm{G}(1, n+1)$.

Let us now consider the infinite groups $\mathrm{G}(2, 2n)$ for $n \geqq 1$. We have proved that $\langle a^2, b^2\rangle$ is normal in $\mathrm{G}(2, 2n)$ with $\mathrm{G}(2,2n)/\langle a^2, b^2\rangle$ isomorphic to D_∞. We now give a result which examines defining relations for $\langle a, b^2\rangle$. This in turn gives information about $\langle a^2, b^2\rangle$.

The relations of $\mathrm{G}(2, 2n)$ are

$$a\underbrace{bb \cdots\cdots bb}_{2n}aba^{-1}a^{-1}b^{-1} = 1, \quad \text{and} \quad b\underbrace{aa \cdots\cdots aa}_{2n}bab^{-1}b^{-1}a^{-1} = 1.$$

We use the modified Todd-Coxeter algorithm to find a presentation for the subgroup $\langle a, b^2\rangle$. See Beetham and Campbell (to appear) for a proof that the algorithm gives a presentation of the subgroup. We obtain the following table as in Campbell (1969), where $x = a$ and $y = b^2$.

$$\begin{aligned}
1.a &= x.1\\
1.b &= 1.1\\
2.b &= y.1\\
2.a &= 1.3\\
3.a &= xy^n x.2\\
3.b &= 1.4\\
4.b &= yx^{2n}.3\\
4.a &= 1.5\\
5.a &= xy^n xy^n.4\\
5.b &= 1.6\\
6.b &= yx^{2n}(xy^n x)^n.5\\
6.a &= 1.7\\
7.a &= (xy^n xy^n)(yx^{2n})^n.6
\end{aligned}$$

....

It is easy to check, using induction, that if k is odd

$$k.a = w(k).(k-1),$$
$$k.b = 1.(k+1),$$

and for k even

$$k.b = w(k).(k-1),$$
$$k.a = 1.(k+1),$$

where $w(k)$ is a word in x and y depending on k.

Now $w(1) = x$, $w(2) = y$, $w(3) = xy^nx$, $w(4) = yx^{2n}$ and by induction we can show that

$$w(k) = w(k-2)[w(k-3)]^n; \qquad k \geqq 5.$$

Each coset gives one relation for the subgroup $\langle x, y\rangle$. Denote by $R(k) = 1$ the relation obtained from coset k. Then

$$R(1) = (xy^nx)^n yxy^{-1}x^{-1},$$
$$R(2) = (yx^{2n})^n xy^n xyx^{-2}y^{-1},$$

and by induction we obtain

$$R(k) = [w(k+2)]^n w(k+1)w(k)[w(k-1)]^{-1}[w(k)]^{-1}; \qquad k \geqq 3.$$

Notice that if $n \leqq 1$ precisely the same argument holds. We have proved the following theorem.

THEOREM 4. *The subgroup* $H = \langle a, b^2\rangle$ *of* $G(2, 2n)$ *has a presentation*

$$H = \langle x, y \mid R(k) = 1, k = 1, 2, 3, \cdots\rangle,$$

where the $R(k)$ *are given inductively as above.*

3. Examples

In this section we examine the groups $G(2, n)$ for $-3 \leqq n \leqq 5$ and also the group $G(3, 1)$.

G(2, −3) This group has the presentation

$$G(2,-3) = \langle a, b \mid ab^2a^{-1} = ba^{-3}b, ba^2b^{-1} = ab^{-3}a\rangle.$$

Let H be the subgroup of $G(2,-3)$ generated by $[a^{-1}, b^{-1}]$, $[a^{-1}, b]$, $[a, b]$. Clearly $H \leqq G'(2,-3)$ and in fact $|G(2,-3) : H| = 18$ so H is a subgroup of index 2 in $G'(2,-3)$. Put $x = [a^{-1}, b^{-1}]$, $y = [a^{-1}, b]$, $z = [a, b]$. Using the modification of the Todd-Coxeter coset enumeration algorithm we obtain a presentation for H on the generators x, y, z with the following relations:

$x^{-2}yz^{-1}y^{-1}x^2y^{-1}zy = 1,$

$x^{-1}yzx^{-1}y^{-1}z = 1,$

$yz^{-1}y^{-1}x^2zy^{-1}xy^{-1}x = 1,$

$x^{-3}yzy^{-1}xy^{-1}x^{-1}yz^{-1}x^{-1}y^2 = 1,$

$x^{-2}yz^{-1}y^{-1}x^2z^{-1}y^{-1}xzy^{-1}xyz^{-1}y = 1,$

$xy^{-1}zy^{-2}x^2yx^{-1}yzy^{-1} = 1,$

$y^2xzy^{-1}x^2y^{-2}xzyx^{-1}yzxy^{-1}z = 1,$

$yx^{-1}yz^{-1}x^{-1}yzy^{-1}z^{-1}x^{-1}y^2x^{-2}yx^{-1}yz^{-1}x^{-1}yzy^{-2}xy^{-1}x^{-1}yz^{-1}y^{-1}x = 1,$

$y^{-1}z^2y^{-1}x^{-1}yz^{-1}x^{-1}yz = 1,$

$x^{-3}yz^{-1}x^{-1}y^2xy^{-1}zy^{-1}x^{-1}y = 1,$

$x^{-1}yx^{-1}yzy^{-1}x^{-1}yz^{-2}yx^{-1}yzy^{-2}zy^{-1}z^{-1}y = 1,$

$x^{-1}yx^{-1}yz^{-2}yx^{-1}y^2z^{-2}y^{-1}xy^{-1}x^{-2}y = 1,$

$xy^{-2}xzyz^{-1}y^{-1}xzy^{-1}xy^{-1}x^2y^{-2}xzyz^{-1}yx^{-1}yzy^{-1} = 1,$

$y^{-1}xy^{-1}xy^{-1}zy^{-1}x^{-1}yz^{-1}yz^{-1}y^{-1}xzy^{-1}xy^{-1}x^2z^{-1}y^{-1}xzy^{-1}xy^{-1}x^2y^{-2}xzy^{-1}$
$\quad xy^{-1}x^2y^{-2}xz = 1.$

It would not be a too tedious task to produce these defining relations by hand. We however used a programme Wilde (1967) to find the words in the subgroup generators which give the relations between the coset representatives. We wrote a programme to find from these words a presentation for the subgroup H. Twenty relations were obtained and the shorter relations used to simplify the longer ones until the programme Beetham (unpublished) could handle the coset enumeration. During the simplifications six of the rrelations were found to be redundant.

The index of $\langle x\rangle$ in H is 32 and $\langle x\rangle$ is not a normal subgroup of H. Hence H is not abelian and since $H \leqq G'(2, -3)$ the group $G(2, -3)$ is not metabelian. In fact $|H| = 2^7.3$ and so $|G(2, -3)| = 2^8.3^3$. By Theorem 3 $G(2, -3)$ has a homomorphic image isomorphic to $G(1, 3)$, a group of order 27, and so $G(2, -3)$ is an extension of a 2-group by $G(1, 3)$. Since $|G(2, -3) : \langle a\rangle| = 2^6.3$ the order of a is 36. Hence a^4 has order 9 and so a^4 is contained in a Sylow 3-subgroup P. If $G(2, -3)$ is nilpotent P is normal in $G(2, -3)$. Let N be the normal closure of $\langle a^4\rangle$ in $G(2, -3)$. Since P is normal $N \leqq P$ so $G(2, -3)/N$ has order divisible by 2^8. However $|G(2, -3)/N| = 24$, so $G(2, -3)$ is not nilpotent.

Next we look at $G(2, 3)$, the structure of which is similar to $G(2, -3)$.

$G(2, 3)$ This group has the presentation

$$G(2,3) = \langle a, b \mid ab^3a = ba^2b^{-1}, ba^3b = ab^2a^{-1} \rangle.$$

It is easy to deduce the relations $(ab)^3 = 1$ and $(ba)^3 = 1$ which are of some help in the coset enumerations.

Let $H = \langle [a,b], [a^{-1},b^{-1}], [a^{-1},b] \rangle$. Then H is a subgroup of index 18 in $G(2,3)$ and has index 2 in $G'(2,3)$. With $x = [a,b]$, $y = [a^{-1},b^{-1}]$, $z = [a^{-1},b]$ we obtain a presentation of H on x, y, z with eight relations

(1) $$[x^2, y^2] = 1,$$

(2) $$[y^2, z^2] = 1,$$

(3) $$zyx^{-1}z^{-1}yx^{-1} = 1,$$

(4) $$(yx)^2z^{-2} = 1,$$

(5) $$y^{-3}zx^{-1}z^{-1}x^2y^2z^{-1}x^2y^{-1}zx^{-1} = 1,$$

(6) $$y^2z^{-1}x^2y^{-1}zx^2y^2z^{-1}x^2yzx^2 = 1,$$

(7) $$z^{-1}y^{-2}zx^{-3}y^{-2}z^{-1}y^{-2}zx^{-1}y^{-2} = 1,$$

(8) $$x^{-1}z^{-1}y^3z^{-1}y^2xz^{-1}yx^{-2}z = 1.$$

The same techniques as used for $G(2,-3)$ were used to find presentations of subgroups of $G(2,3)$.

The subgroup $\langle x^2, y^2, z^2 \rangle$ is abelian. For, from (3) $[z^2, yx^{-1}] = 1$, from (4) $[z^2, yx] = 1$ and so $[z^2, x^2] = 1$. The relations of H are sufficiently complicated to make it very difficult to find a presentation of a subgroup L of H of index greater than two by the modified Todd-Coxeter algorithm. However if the subgroup of H is known to be abelian the simplification is substantial and so it might be reasonable to try to find a presentation for $\langle x^2, y^2, z^2 \rangle$. The index $|H: \langle x^2, y^2, z^2 \rangle|$ is 256 which the programme Wilde (1967) cannot handle. We reduce the problem by finding a subgroup K of index 2 in H and a larger abelian subgroup than $\langle x^2, y^2, z^2 \rangle$.

Let $A = \langle x^2, y^2, z^2, yx^2y, yz^2y \rangle$. Then A is abelian. This can be checked using the relations (1) to (4). We know x^2, y^2 and z^2 commute. To show $[x^2, yx^2y] = 1$ use (4) to obtain $[x^2, yxyx] = 1$. Since x^2 commutes with yxy^{-1}, $[x^2, yx^2y^{-1}] = 1$ and so x^2 commutes with yx^2y as required. Next we show that $[z^2, yx^2y] = 1$. Use (3) to obtain $[z^2, yx^{-1}] = 1$. But $[z^2, yx^{-1}] = 1$ gives both $[z^2, xy^{-1}] = 1$ and $[z^2, yx] = 1$ so $[z^2, yx^2y^{-1}] = 1$ and the result follows. Now $[z^2, yx^2y] = 1$ implies $[x^2, yz^2y] = 1$ and it remains to show that $[z^2, yz^2y] = 1$. But $yz^2y = y^2xyxy$ by (4) and z^2 commutes with xy since we have shown above that $[z^2, xy^{-1}] = 1$. Hence A is abelian.

Let $K = \langle x^2, y, z \rangle$. Then $H \geqq K \geqq A$ and $|H:K| = 2$. With $r = x^2$, $s = y$, $t = z$ we can find a presentation of K on r, s, t with the fourteen relations which may be simplified, using the fact that A is abelian, to the following:

$$
\begin{aligned}
[r, t^2] &= 1,\\
[r, srs] &= 1,\\
[r, st^2s] &= 1,\\
[r, s^2] &= 1,\\
[s^2, t^2] &= 1,\\
[r, ts^2t] &= 1,\\
t^{-2}(srs^{-1}t^{-1}r)^2 &= 1,\\
strs^2t^{-1}rs^{-1}trs^2t^{-1}r &= 1,\\
s^{-3}t^{-1}stst^{-1}st &= 1,\\
tsr^{-1}t^{-2}s^{-1}r^{-1}t^{-1}srt^{-2}s^{-1}r^{-1} &= 1,\\
t^{-1}srsts^{-1}tsrst^{-1}rsr &= 1,\\
t^{-1}s^{-2}t^{-1}r^{-1}st^{-1}st^{-2}sts^{-1}r^{-1} &= 1,\\
s^{-1}ts^{-1}t^2r^{-1}s^{-2}t^{-1}s^{-1}tsr^{-1}t &= 1,\\
s^3t^{-1}rst^{-1}s^{-1}t^{-1}rst^{-1} &= 1.
\end{aligned}
$$

Now $A = \langle r, s^2, t^2, srs, st^2s\rangle$ and $|K : A| = 32$. The modified Todd-Coxeter algorithm applied to A as a subgroup of K gives a presentation for the abelian group A from which it can be shown that $|A| = 768$ and $r^{12} = (s^2)^{12} = (t^2)^{12} = (srs)^{12} = (st^2s)^4 = 1$. This shows that the order of $G(2,3)$ is $2^{15}.3^3$. Note that H is not abelian since $\langle x\rangle$ is not normal in H and so $G(2,3)$ is not metabelian. To show $G(2,3)$ is not nilpotent use the fact that ab is an element of order 3, so if $G(2,3)$ is nilpotent ab is contained in the normal Sylow 3–subgroup. This is clearly impossible since a and b have order 72.

To complete a study of the finite groups $G(2, n)$ for $-3 \leqq n \leqq 5$ we must examine $G(2,-1)$, $G(2,1)$ and $G(2,5)$. $G(2,-1)$ is trivial by Theorem 3 and $G(2,1)$ is proved to be trivial in Example 2 of Campbell (1969).

$G(2,5)$ The group $G(2,5)$ is isomorphic to $G(1,5)$. To show this we must prove that $a^5 = b^{-5}$. The subgroup $\langle a\rangle$ is normal in $G(2,5)$ and $G(2,5)/\langle a\rangle \simeq Z_5$. Hence $\langle a^5\rangle = \langle b^5\rangle = G'(2,5)$. Since $a^5, b^5 \in Z(G(2,5))$, the centre of $G(2,5)$, we have

$$a^{-2}ba^2 = b^6, \qquad b^{-2}ab^2 = a^6.$$

Then $a^{-2}b^5a^2 = b^{30}$ so $b^{25} = 1$. Also $b^{-1}a^{-2}b = b^5a^{-2}$ gives $b^{-1}a^{-4}b = b^{10}a^{-4}$, so $b^{-1}ab = b^{10}a$. Thus $b^{-2}ab^2 = b^{20}a$ and this together with $b^{-2}ab^2 = a^6$ gives $a^5 = b^{-5}$.

Next consider the three infinite groups $\mathbf{G}(2,-2)$, $\mathbf{G}(2,2)$ and $\mathbf{G}(2,4)$.

$\mathbf{G}(2,2)$ We prove $\mathbf{G}(2,2)$ is isomorphic to $\mathbf{D}_\infty$. This follows from Theorem 2 once we have shown $a^2 = b^2 = 1$. Now

$$\mathbf{G}(2,2) = \langle a, b \mid ab^2a^{-1} = ba^2b,\ ba^2b^{-1} = ab^2a\rangle.$$

But $ab^2a^{-1} = ba^2b^{-1}b^2 = ab^2ab^2$ and so $a^{-2} = b^2$. Then $ab^2a^{-1} = ba^2b = 1$ and so $b^2 = 1$. Since $a^{-2} = b^2$ this gives $a^2 = 1$.

$\mathbf{G}(2,-2)$ We show that $\mathbf{G}(2,-2)$ is also isomorphic to $\mathbf{D}_\infty$. Use the relations of Theorem 4 to give a presentation for $H = \langle a, b^2\rangle$.

$$H = \langle x, y \mid R(k) = 1,\ k = 1, 2, 3, \cdots\rangle.$$

The first four of these relations are

$$\begin{aligned}
&x^{-2}yx^{-1}yxy^{-1} = 1,\\
&x^2y^{-1}xy^{-1}xyx^{-2}y^{-1} = 1,\\
&yx^{-1}yx^{-1}yx^{-1}y^{-1}xy^{-1}x^{-1}yx^{-1} = 1,\\
&xy^{-1}x^3y^{-1}xy^{-1}x^{-2}yxy^{-1} = 1.
\end{aligned}$$

It is straightforward to deduce from these relations that $y = 1$ and $x^2 = 1$. H is isomorphic to the cyclic group of order 2 since the relations $R(k) = 1$ hold in the group $\langle x, y \mid y = 1, x^2 = 1\rangle$ for $k \geqq 5$. This gives $a^2 = 1$ and therefore $b^2 = 1$. Thus $\mathbf{G}(2,-2)$ is isomorphic to the infinite dihedral group by Theorem 2.

$\mathbf{G}(2,4)$ Again use Theorem 4. H can easily be shown to have only five relations. For, $w(k) = w(k-2)[w(k-3)]^2$, $k \geqq 5$, and

$$[w(k+2)]^2w(k+1)w(k)[w(k-1)]^{-1}[w(k)]^{-1} = 1, \qquad k \geqq 3.$$

Then $w(k+2) = w(k)[w(k-1)]^2$ and so $w(k+2)w(k+1)w(k)w(k-1) = 1$, $k \geqq 3$. Then $w(k+3) = w(k-1)$, $k \geqq 3$. Hence, for $n = 2$, H has a presentation

$$H = \langle x, y \mid R(1) = R(2) = R(3) = R(4) = R(5) = 1\rangle.$$

This presentation simplifies to

$$H = \langle x, y \mid y^{-1}xy = x^{-3},\ xy^2x^{-1} = y^{-2},\ yx^{-1}y^{-1} = x^{-5}\rangle,$$

and so H is a group of order 64. Hence $\langle a^2, b^2\rangle$ is a group of order 32 whose structure is easily exhibited. $\mathbf{G}(2,4)$ is an extension of this group of order 32 by $\mathbf{D}_\infty$.

$\mathbf{G}(3,1)$ This group has the presentation

$$\mathbf{G}(3,1) = \langle a, b \mid ab^3 = b^2aba,\ ba^3 = a^2bab\rangle.$$

The order of $\mathbf{G}(3,1)$ is 120. For, using the modified Todd-Coxeter algorithm on the subgroup $\langle a\rangle$ gives $|\mathbf{G}(3,1) : \langle a\rangle| = 12$ and the presentation for $\langle a\rangle$ obtained is $\langle a \mid a^{10} = 1\rangle$.

$SL(2, 5)$ is generated by the two matrices A and B where

$$A = \begin{pmatrix} 4 & 0 \\ 4 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 & 3 \\ 0 & 4 \end{pmatrix}.$$

But $AB^3 = B^2ABA$ and $BA^3 = A^2BAB$ and so $SL(2, 5)$ is a homomorphic image of $G(3, 1)$. However $|SL(2, 5)| = 120$ and so $G(3, 1)$ is isomorphic to $SL(2, 5)$.

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Mathematical Institute
University of St. Andrews
St. Andrews, Fife
Scotland

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10.—Applications of the Todd-Coxeter Algorithm to Generalised Fibonacci Groups.* By C. M. Campbell and E. F. Robertson,

Mathematical Institute, University of St Andrews. *Communicated by* Dr T. S. Blyth

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Reference (13)

Synopsis

We use programmes for the Todd-Coxeter coset enumeration algorithm and the modified Todd-Coxeter coset enumeration algorithm to investigate a class of generalised Fibonacci groups. In particular we use these techniques to discover a finite non-metacyclic Fibonacci group and to study its structure.

1. Introduction

Let F_n be the free group on $\{a_i: i \in \mathbb{Z}_n\}$ and let ϕ be the permutation $(0, 1, 2, ..., n-1)$. Denote by θ the automorphism of F_n induced by ϕ, namely $a_i\theta = a_{i\phi}$. For r and k integers, $r \geqq 1$ and $k \geqq 0$, let ω be the word in F_n

$$\omega = a_0 a_1 \dots a_{r-1} a_{r+k-1}^{-1}.$$

Define the generalised Fibonacci group $F(r, n, k)$ to be the group $F_n/N(\omega)$, where $N(\omega)$ denotes the normal closure of $\{\omega\theta^{i-1}: 1 \leqq i \leqq n\}$ in F_n. The Fibonacci groups $F(r, n, 1)$ have already received attention by several mathematicians, see for example [6]. In this note we use computer techniques both to solve some problems relating to Fibonacci groups and to investigate the generalised Fibonacci groups.

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2. The Groups $F(r, n, k)$

If $k_1 \equiv k_2 \bmod n$, then $F(r, n, k_1) \cong F(r, n, k_2)$ so that when we write $F(r, n, k)$ we shall assume that k has been reduced mod n. To facilitate a systematic computer study of generalised Fibonacci groups, use was made of the following two isomorphisms.

Lemma 1. $F(r, n, k) \cong F(r, n, 1-r-k)$.

Proof. Let $F(r, n, k)$ be generated by $\{a_i: i \in \mathbb{Z}_n\}$ and $F(r, n, 1-r-k\}$ by $\{b_i: i \in \mathbb{Z}_n\}$. If F_n is the free group on $\{a_i: i \in \mathbb{Z}_n\}$, then the map $\phi: F_n \to F(r, n, 1-r-k)$ induced by $a_i\phi = b_{-i}^{-1}$ gives an isomorphism between $F(r, n, k)$ and $F(r, n, 1-r-k)$.

We state the second isomorphism without proof since its proof is technical and somewhat lengthy. However, we give a proof of this result and some other results on metacyclic generalised Fibonacci groups in [3] and [4].

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THEOREM 2. *If $r \equiv 1 \bmod n$ and k_1 and k_2 are coprime to n,*

$$F(r, n, k_1) \cong F(r, n, k_2)$$

and is metacyclic of order $r^n - 1$.

The following lemma whose proof is elementary enables some generalised Fibonacci groups to be easily described.

LEMMA 3. (i) *$F(r, 1, k) \cong C_{r-1}$, the cyclic group of order $r-1$,*

(ii) *$F(1, n, k) \cong F_d$, the free group of rank d, where $d = h.c.f.\,(n, k)$,*

(iii) *$F(r, n, 0) \cong C_{r-1}$ if $h.c.f.\,(r-1, n) = 1$; otherwise $F(r, n, 0)$ is infinite.*

It is easy to see that $F(r, n, k)$ is a homomorphic image of $F(r, \lambda n, k)$ for any positive integer λ. A further result on homomorphic images is given by the following theorem.

THEOREM 4. *For n coprime to 6, $F(2, n, -\frac{1}{2})$ is a perfect group, and has $SL(2, p)$ as a homomorphic image for any prime divisor p of n.*

Proof. The relations of $F(2, n, -\frac{1}{2})$ are

$$a_0a_1 = a_{1/2},\ a_1a_2 = a_{3/2},\ a_2a_3 = a_{5/2},\ \ldots,\ a_{n-1}a_0 = a_{-1/2}.$$

Now $a_{1/2}a_{3/2} = a_1$ and thus $(a_0a_1)(a_1a_2) = a_1$. In the factor of $F(2, n, -\frac{1}{2})$ by its derived group both $a_0a_1a_2 = 1$ and $a_1a_2a_3 = 1$ and thus $a_0 = a_3$. Since n is coprime to 3 this proves $F(2, n, -\frac{1}{2})$ is perfect.

If $x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$, where the entries in the matrices are in $GF(p)$, it is well known that x and y generate $SL(2, p)$. Putting $a_i = x^{-i}y^{-1}x^{i+\frac{1}{2}}$, it may be checked that $a_0, a_1, \ldots, a_{p-1}$ satisfy the relations for $F(2, p, -\frac{1}{2})$.

Theorem 8 of [6] shows that any finite group with an n-generator n-relation presentation has factor by derived group of rank at most 3. This result enables us to show that the generalised Fibonacci groups $F(2\lambda r-1, \mu r, 1-\nu r)$ are infinite if $r \geqq 4$, for any positive integers λ, μ, ν. This follows since adding the relations $a_i^2 = 1$, $0 \leqq i \leqq \mu r - 1$ shows the factor by the derived group to have rank at least 4.

3. SOME RESULTS USING COSET ENUMERATION

The Todd-Coxeter coset enumeration algorithm is described in [5] and this algorithm has been programmed by Dr M. J. Beetham for the IBM 360 computer of the University of St Andrews. This programme enables the index of a finitely generated subgroup in a finitely generated finitely presented group to be obtained, enumerating up to $32\,000/(2n+1)$ cosets, where n is the number of generators of the group. Using this programme, we have shown that $F(3, 6, 1)$ is a finite non-metacyclic group, being the first, and as far as we know the only, finite non-metacyclic Fibonacci group so far discovered. $F(3, 6, 1)$ may be reduced to a 2-generator group with presentation

$$\langle a, b \mid a^2b^{-1}ab^{-1}a^2b^{-1}ab = b^2a^{-1}baba^2 = 1 \rangle.$$

The programme shows $|F(3, 6, 1)| = 1512 = 2^3.3^3.7$ and that a and b have order 84. The homomorphic image obtained by adding $[a^{12}, b] = 1$ also has order 1512, and so a^{12} generates a unique Sylow 7-subgroup in the centre of $F(3, 6, 1)$. The Sylow 3-subgroup is normal, elementary abelian, and may be generated by a^{28}, a^2b^{18} and $a^4b^{-1}a^2b^{-1}$. There are 9 Sylow 2-subgroups isomorphic to the quaternion group and one may be generated by a^{21} and $(b^{-1}ab^{-1}a^2b^{-1})^{21}$. Although soluble $F(3, 6, 1)$ by the above is obviously not nilpotent.

We have made a systematic search for finite generalised Fibonacci groups for $r, n, k \leqq 6$. Apart from the finite groups given by the results of section 2 and $F(3, 6, 1)$ discussed above, we have obtained the following orders:

$$|F(6, 3, 2)| = 5, \quad |F(2, 6, 2)| = 7, \quad |F(4, 4, 2)| = 39,$$

$$|F(3, 3, 2)| = 48, \quad |F(2, 5, 2)| = 120, \quad |F(6, 4, 2)| = 125.$$

$F(6, 3, 2)$ and $F(2, 6, 2)$ are clearly cyclic, $F(4, 4, 2)$ and $F(6, 4, 2)$ are metacyclic, $F(3, 3, 2)$ is not metacyclic, having the binary tetrahedral group, see for example [5], section 6.5, as derived group. $F(2, 5, 2)$ must be isomorphic to $SL(2, 5)$ by Theorem 4.

In answer to a question posed in [6] we have shown that the homomorphic image of $F(2, 10, 1)$ obtained by adding the relation $a_0a_5^{-1} = 1$ is a metacyclic group of order 253.

4. Infinite $F(r, n, k)$

A modification of the Todd-Coxeter coset enumeration algorithm is described in [1] and has been programmed by Wilde in [7]. Using this programme together with one of our own, see [2], we are able to show that the groups $F(5, 5, 3)$, $F(3, 5, 4)$ and $F(3, 6, 5)$ are infinite.

$F(5, 5, 3)$ may be reduced to a 2-generator group with the presentation

$$\langle a, b \mid bab^{-2}ababa^{-2}ba = a^{-1}bababa^{-1}b^{-1}a^{-1}b^{-1} = 1\rangle.$$

The subgroup $H = \langle (ab)^2, (ba)^2\rangle$ has index 4. With $x = (ab)^2$, $y = (ba)^2$, H has the presentation

$$\langle x, y \mid x^5 = y^5 = (xy)^3\rangle.$$

This is the binary polyhedral group $\langle 5, 5, 3\rangle$ discussed in [5], section 6.5, where it is shown to be infinite.

$F(3, 5, 4)$ has the following presentation as a 2-generator group:

$$\langle a, b \mid b^{-1}ab^{-1}a^{-1}bab^{-1}aba^{-1} = b^{-1}a^{-1}bab^{-1}a^{-1}b^{-1}aba^{-1} = 1\rangle.$$

The subgroup $H = \langle a, b^2\rangle$ has index 10 in $F(3, 5, 4)$ and with $x = a$, $y = b^2$ it has the presentation

$$\langle x, y \mid yxy^{-1}x^{-1}yxy^{-1}x^{-1}y^{-1}xyx^{-1}y^{-1}xyx^{-1}$$
$$= y^{-1}x^{-1}yx^{-1}y^{-1}xyx^{-1}y^{-1}xyxy^{-1}x^{-1}yx = 1\rangle.$$

H has a free abelian factor of rank 2 and so $F(3, 5, 4)$ is infinite.

$F(3, 6, 5)$ has the following presentation as a 2-generator group:

$$\langle a, b \mid b^{-1}ab^{-1}a^{-1}bab^{-1}a^{-1}b^{-1}aba^{-1} = b^{-1}a^{-1}bab^{-1}a^{-1}ba^{-1}b^{-1}aba^{-1} = 1\rangle.$$

The subgroup $H = \langle a^2, b^2, a^{-1}b^2a, b^{-1}a^2b\rangle$ has index 24 in $F(3, 6, 5)$ and a similar method enables us to show that H has an infinite cyclic homomorphic image and thus $F(3, 6, 5)$ is infinite.

Finally, Theorem 4 together with the fact that $F(2, 5, 2) \cong SL(2, 5)$ poses the question as to whether $F(2, p, -1/2) \cong SL(2, p)$, p a prime, $p \geqq 7$. However, the modified Todd-Coxeter coset enumeration algorithm may be used to show that this is not the case, $F(2, p, -1/2)$ being infinite, $p \geqq 7$.

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A NOTE ON FIBONACCI TYPE GROUPS

BY
C. M. CAMPBELL AND E. F. ROBERTSON

1. Introduction. Let F_n be the free group on $\{a_i : i \in \mathbb{Z}_n\}$ where the set of congruence classes mod n is used as an index set for the generators. The permutation $(1, 2, 3, \ldots, n)$ of $\mathbb{Z}_n$ induces an automorphism θ of F_n by permuting the subscripts of the generators. Suppose w is a word in F_n and let $N(w)$ denote the normal closure of $\{w\theta^{i-1} : 1 \leq i \leq n\}$. Define the group $G_n(w)$ by $G_n(w) = F_n/N(w)$ and call $w\theta^{i-1} = 1$ the relation (i) of $G_n(w)$.

In this note we consider the group $G_n(w)$ where w is the word

$$w = a_h a_{2h} \cdots a_{rh}(a_{rh+k}^{-1})$$

and r, h, k are integers such that $k \geq 0$, $h \geq 1$, $r \geq 2$. For this particular choice of w we denote $G_n(w)$ by $\mathrm{R}(r, n, k, h)$. The groups $\mathrm{R}(2, n, n-1, 2)$ are discussed in [6] while the groups $\mathrm{R}(2, n, k, h)$ have been investigated by Johnson and Mawdesley. The groups $\mathrm{R}(r, n, k, 1)$ are the *generalized Fibonacci groups* $\mathrm{F}(r, n, k)$ discussed in [2], [3], [4] and [7] while the groups $\mathrm{R}(r, n, 1, 1)$ are the ordinary *Fibonacci groups* $\mathrm{F}(r, n)$ discussed in [5] and [8]. We exhibit some isomorphisms, showing that more of the groups $\mathrm{R}(r, n, k, h)$ are generalized Fibonacci groups than are indicated above. We also discuss the group $\mathrm{R}(3, 6, 5, 2)$, a finite non-metacyclic group which is not a generalized Fibonacci group.

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2. Some isomorphisms. It follows immediately from the definition that if $k_1 \equiv k_2 \bmod n$ and $h_1 \equiv h_2 \bmod n$ then $\mathrm{R}(r, n, k_1, h_1) \cong \mathrm{R}(r, n, k_2, h_2)$ so that when we write $\mathrm{R}(r, n, k, h)$ we shall assume that k and h have been reduced mod n.

LEMMA 1.

$$\begin{aligned}\mathrm{R}(r, n, k, h) &\cong \mathrm{R}(r, n, k+(r-1)h, -h)\\ &\cong \mathrm{R}(r, n, -k, -h)\\ &\cong \mathrm{R}(r, n, -k-(r-1)h, h).\end{aligned}$$

Proof. The isomorphisms are immediate on considering the maps ϕ_1, ϕ_2, ϕ_3 from the free group F_n on $\{x_i : i \in \mathbb{Z}_n\}$ to $\mathrm{R}(r, n, k, h)$ induced by $x_i\phi_1 = a_i^{-1}$, $x_i\phi_2 = a_{-i}$ and $x_i\phi_3 = a_{-i}^{-1}$.

LEMMA 2. *If α is an integer coprime to n then*

$$\mathrm{R}(r, n, k, h) \cong \mathrm{R}(r, n, k/\alpha, h/\alpha).$$

Proof. This isomorphism follows from considering the map ϕ from the free group on $\{x_i : i \in \mathbb{Z}_n\}$ to $\mathrm{R}(r, n, k, h)$ induced by $x_i\phi = x_{i/\alpha}$.

Notice that it follows from this result that if h is coprime to n, $\mathrm{R}(r, n, k, h) \cong \mathbf{F}(r, n, k/h)$.

THEOREM 3. *Suppose that* $(r-1)h \equiv 0 \bmod n$ *and* k *is coprime to* n, *then*

$$\mathrm{R}(r, n, k, h) \cong \mathrm{F}(r^{(n,h)}, d, \gamma)$$

where $d = n/(n, h)$ *and* γ *is such that* $(n, h) = \beta n + \gamma h$.

Proof. By lemma 2 we can assume without loss of generality that $k=1$. The first relation of $\mathrm{R}(r, n, 1, h)$ reduces to

$$(a_h a_{2h} \cdots a_{dh})^{(r-1)/d} a_h = a_{h+1}$$

where the generators $a_h, a_{2h}, \ldots, a_{dh}$ are distinct. This allows us to express a_{h+1} in terms of $a_h, a_{2h}, \ldots, a_{dh}$ and relation (ih) allows us to express $a_{(i+1)h+1}$ also in terms of $a_h, a_{2h}, \ldots, a_{dh}$ for $1 \leq i \leq d-1$. Substituting these expressions in relation (2) gives

$$(a_h a_{2h} \cdots a_{dh})^{(r^2-1)/d} a_h = a_{h+2}.$$

Continuing in this way we obtain

$$(a_h a_{2h} \cdots a_{dh})^{(r^j-1)/d} a_h = a_{h+j}, \qquad 1 \leq j \leq (n, h),$$

since a_{h+j}, $1 \leq j \leq (n, h)$ are distinct and $a_{h+(n,h)} \in \{a_h, a_{2h}, \ldots, a_{nh}\}$. At this stage the n relations for $\mathrm{R}(r, n, 1, h)$ have been reduced to the d relations

$$((a_h a_{2h} \cdots a_{dh})^{(r^{(n,h)}-1)/d} a_h a_{h+(n,h)}^{-1})\theta^{(i-1)h} = 1, \qquad 1 \leq i \leq d.$$

Putting $x_i = a_{ih}$, $1 \leq i \leq d$ we obtain the relations

$$((x_1 x_2 \cdots x_d)^{(r^{(n,h)}-1)/d} x_1 x_{1+\gamma}^{-1})\bar{\theta}^{i-1} = 1, \qquad 1 \leq i \leq d,$$

where $\bar{\theta}$ permutes the subscripts of x_i, $1 \leq i \leq d$, according to the permutation $(1, 2, \ldots, d)$. The result now follows.

COROLLARY. *With the conditions on* r, n, k, h *as in the statement of Theorem* 3, $\mathrm{R}(r, n, k, h)$ *is metacyclic of order* $r^n - 1$.

Proof. This follows from Theorem 1 of [3] and Theorem 3 on showing that $r^{(n,h)} \equiv 1 \bmod d$ and γ is coprime to n. These are straightforward applications of elementary number theory.

Notice, using the results of [4], that if $\mathrm{R}(r, n, k_1, h_1)$ and $\mathrm{R}(r, n, k_2, h_2)$ satisfy the conditions of the above theorem then they are isomorphic if, and only if, $(n, h_1) = (n, h_2)$.

Next we show that if $(n, k, h) \neq 1$, then $\mathrm{R}(r, n, k, h)$ is infinite.

THEOREM 4. *If* $(n, k, h) = d \neq 1$, *then*

$$\mathrm{R}(r, n, k, h) \cong \mathop{*}_{d} \mathrm{R}(r, n/d, k/d, h/d),$$

the free product of d *copies of* $\mathrm{R}(r, n/d, k/d, h/d)$.

Proof. Let $\alpha=n/d$, $\beta=k/d$, $\gamma=h/d$ and fix t with $0\leq t\leq d-1$. With $x_j=a_{jd+t}$ the relations $(id+t)$, $1\leq i\leq \alpha$, reduce to

$$(x_\gamma x_{2\gamma}\cdots x_{r\gamma}x^{-1}_{r\gamma+\beta})\bar{\theta}^{i-1}=1,\qquad 1\leq i\leq\alpha,$$

where the subscripts of the x_i are reduced mod α and permuted by $\bar{\theta}$ according to the permutation $(1, 2, \ldots, \alpha)$. The result now follows.

3. The group R(3, 6, 5, 2). The only Fibonacci group known to be finite and not metacyclic is F(3, 6), a group of order 1512, see [2], where the three known finite non-metacyclic generalized Fibonacci groups are discussed. The only finite non-metacyclic group which we have discovered in the class $R(r, n, k, h)$ other than these generalized Fibonacci groups is R(3, 6, 5, 2).

Using Tietze transformations the following 2-generator, 2-relation presentation is obtained.

$$R(3, 6, 5, 2)=\langle a, b \mid a^{-1}ba^2b^{-1}ab^2=(ba^{-1}b^{-1}a^{-1})^2ba^{-1}bab^{-1}a=1\rangle.$$

We have investigated this group using the coset enumeration programme [1] which shows that $|R(3, 6, 5, 2)|=1512=2^3\cdot 3^3\cdot 7$. It is soluble but not metabelian and has the following Sylow structure. A Sylow 2-subgroup is cyclic and generated by a. It is not normal. Both the Sylow 3-subgroup and the Sylow 7-subgroup are normal, the Sylow 3-subgroup being the non-abelian group of order 27. Despite the coincidence in the orders R(3, 6, 5, 2) is not isomorphic to F(3, 6) since, for example, F(3, 6) has Q_8 as a Sylow 2-subgroup.

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MATHEMATICAL INSTITUTE,
UNIVERSITY OF ST. ANDREWS,
ST. ANDREWS, KY16 9SS,
SCOTLAND.

ON A CLASS OF FINITELY PRESENTED GROUPS OF FIBONACCI TYPE

Reference (15)

C. M. CAMPBELL AND E. F. ROBERTSON

G. A. Miller in [10] introduced a class of groups $G_n(\omega)$ defined as follows. Let F_n be the free group on $\{a_i: i \in \mathbb{Z}_n\}$ where the set of congruence classes mod n is used as an index set for the generators. The permutation $(1\ 2\ 3\ \dots\ n)$ of $\mathbb{Z}_n$ induces an automorphism θ of F_n by permuting the subscripts of the generators. For a given word ω in F_n, $G_n(\omega) = F_n/N(\omega)$ where $N(\omega)$ is the smallest normal subgroup of F_n containing $\{\omega\theta^{i-1}: 1 \leqslant i \leqslant n\}$. We shall call the relation $\omega\theta^{j-1} = 1$ the jth relation. Various classes of $G_n(\omega)$ have been studied in detail. For instance the *Fibonacci groups* $\mathrm{F}(r, n)$ are obtained when ω is the word

$$\omega = a_1 a_2 \dots a_r a_{r+1}^{-1},$$

r an integer $r \geqslant 2$; see for example [9]. When k, r are integers, $k \geqslant 0$, $r \geqslant 2$ and ω is the word

$$\omega = a_1 a_2 \dots a_r a_{r+k}^{-1}$$

then the *generalised Fibonacci groups* $\mathrm{F}(r, n, k)$ are obtained; see for example [3], [4] and [5].

In this paper we consider the groups $\mathrm{H}(r, n, s)$, the subclass of $G_n(\omega)$ obtained by taking

$$\omega = a_1 a_2 \dots a_r (a_{r+1} a_{r+2} \dots a_{r+s})^{-1}$$

where r, s are integers, $r > s \geqslant 1$. Notice that the class of Fibonacci groups is a subclass of the class $\mathrm{H}(r, n, s)$ since $F(r, n)$ is isomorphic to $\mathrm{H}(r, n, 1)$. In section 1 we discuss some elementary properties of $\mathrm{H}(r, n, s)$ including a study of when the commutator quotient is finite, this generalising Corollary 2 of Theorem 2 of [7]. In section 2 we obtain results concerning metacyclic groups in the class $\mathrm{H}(r, n, s)$. In [9] two questions are posed relating to the Fibonacci groups $\mathrm{F}(r, n)$ when $r \equiv 1 \bmod n$, namely to find the orders of these groups and also 2-generator 2-relation presentations for them. These two problems were solved in [4] and [5] respectively. The solutions to these two problems may also be obtained as special cases of the more general results of section 2. The main tools we use in investigating metacylic $\mathrm{H}(r, n, s)$ are the Todd–Coxeter coset enumeration algorithm; see for example [6] and the modification to the algorithm described in [1].

1. *Elementary results*

LEMMA 1. *Suppose* $r \equiv 0 \bmod n$ *or* $s \equiv 0 \bmod n$; *then*

(i) $\mathrm{H}(r, n, s) \cong \mathbb{Z}_{r-s}$ *if* $r-s$ *is coprime to* r,

(ii) $\mathrm{H}(r, n, s)$ *is infinite if* $r-s$ *is not coprime to* n.

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Proof. Since $H(r, n, s) \cong F(r-s+1, n, 0)$ if $r \equiv 0 \bmod n$ or $s \equiv 0 \bmod n$, then, as in Lemma 3 (iii) of [3], the result is straightforward.

LEMMA 2. *$H(r, n, s)$ has $H(r, m, s)$ as a homomorphic image if m divides n.*

Proof. Add the relations $a_i a_{i+m}^{-1} = 1$, $i \in \mathbb{Z}_n$ to the relations for $H(r, n, s)$.

LEMMA 3. *$H(r, n, s)$ is infinite if $(r, n, s) \neq 1$.*

Proof. Let $d = (r, n, s) \neq 1$. By Lemma 2, $H(r, n, s)$ has $H(r, d, s)$ as a homomorphic image but $H(r, d, s)$ is infinite by Lemma 1, since d divides both r and s.

It is easy to see that $H(r, n, s)/H'(r, n, s)$, where $H'(r, n, s)$ is the derived group of $H(r, n, s)$, is infinite if $(r, n, s) \neq 1$. We show that in all other cases the commutator quotient of $H(r, n, s)$ is finite.

THEOREM 1. *$H(r, n, s)/H'(r, n, s)$ is finite if, and only if, $(r, n, s) = 1$.*

Proof. The polynomial associated with $H(r, n, s)$ is $f(x)$ where

$$f(x) \equiv 1+x+x^2+\ldots+x^{r-1}-x^{r+1}-x^{r+2}-\ldots-x^{r+s-1}.$$

By Theorem 2 of [8], $H(r, n, s)$ has an infinite abelian factor precisely when there is an nth root of unity ξ satisfying $f(\xi) = 0$. Clearly $\xi \neq 1$.

$$(1-\xi)f(\xi) = 1-2\xi^r+\xi^{r+s};$$

so $(1-\xi)f(\xi) = 0$ when

$$1-\xi^r = \xi^r-\xi^{r+s}. \tag{1}$$

Hence $|1-\xi^r| = |\xi^r-\xi^{r+s}| = |1-\xi^s|$, $\xi \neq 1$ and so $\xi^r = \xi^s$ or $\xi^r = \xi^{-s}$.
Substituting $\xi^r = \xi^s$ into (1) gives

$$1-2\xi^r+\xi^{r^2} = 0;$$

so $(\xi^r-1)^2 = 0$.
Hence $\xi^r = 1$, $\xi^s = 1$, $\xi^n = 1$, $\xi \neq 1$, giving $(r, n, s) \neq 1$.
Substituting $\xi^r = \xi^{-s}$ into (1) gives

$$2-2\xi^r = 0,$$

which again gives $(r, n, s) \neq 1$.

LEMMA 4. *$H(r, n, s)$ is infinite if*

(i) $r+s \equiv 0 \bmod n$, $n \geqslant 5$,

or

(ii) $n = 4$ *and* $r+s$ *is divisible by* 8.

Proof. If $H(r, n, s)$ is finite it is a group of deficiency zero and so by Theorem 9 of [9] it follows that $H(r, n, s)$ is infinite if the commutator quotient has rank $\geqslant 4$. Consider A, the factor group of the commutator quotient of $H(r, n, s)$ obtained by adding the relations $a_i^2 = 1, 1 \leqslant i \leqslant n$. Since $r+s \equiv 0 \bmod n$, the relations for $\mathrm{H}(r, n, s)$ become in A the single relation

$$a_1 a_2 \dots a_r = a_{r+1} a_{r+2} \dots a_n,$$

that is

$$a_1 a_2 \dots a_{n-1} = a_n.$$

Therefore A is isomorphic to the direct sum of $n-1$ copies of $\mathbb{Z}_2$, and so the commutator quotient has rank $\geqslant 4$ if $n \geqslant 5$.

If, however, $n = 4$ and $r+s$ is divisible by 8, then the relations for $\mathrm{H}(r, 4, s)$ become redundant in A and so A has rank 4.

Apart from those mentioned in the results of this section and in the metacyclic results of section 2, the only other finite groups we have discovered in the class $\mathrm{H}(r, n, s)$ are given by the following result, the proof of which is routine and which we therefore omit.

LEMMA 5. (i) $\mathrm{H}(r, 4, r-1) \cong \mathbb{Z}_5$ *if* $r \equiv 2, 3 \bmod 4$,

(ii) $\mathrm{H}(r, 6, r-1) \cong \mathbb{Z}_{13}$ *if* $r \equiv 3, 4 \bmod 6$.

2. *Metacyclic* H(*r*, *n*, *s*)

THEOREM 2. *Suppose that* $r \equiv s \bmod n$ *and* r *is coprime to* n; *then* $\mathrm{H}(r, n, s)$ *has the following* 2-*generator*, 2-*relation presentation*:

$$\left\langle x, y \mid y^{-1} x^s y = x^r, x^s = \prod_{i=1}^{n} y x^{(n-i-1)(r-s)/n} \right\rangle .$$

Proof. Let $r = nt+k$, $s = nu+k$. Then the relations of $\mathrm{H}(r, n, s)$ are

$$(a_j a_{j+1} \dots a_{j+n-1})^t (a_j a_{j+1} \dots a_{j+k-1}) \big((a_{j+k} a_{j+k+1} \dots a_{j+k-1})^u (a_{j+k} a_{j+k+1} \dots a_{j+2k-1}) \big)^{-1} = 1, 1 \leqslant j \leqslant n.$$

Let $x = a_1 a_2 \dots a_n$, $y = (a_{k+1} a_{k+2} \dots a_k)^u (a_{k+1} a_{k+2} \dots a_{2k})$ and let $K = \langle x, y \rangle$. We use the modified Todd–Coxeter algorithm to show that K has index 1 in $\mathrm{H}(r, n, s)$ and hence to find a presentation for $\mathrm{H}(r, n, s)$ on the generators x and y. We use a similar notation to that described in [2]. Define n cosets of K in $\mathrm{H}(r, n, s)$ as follows: let coset $1 = K$, and for $2 \leqslant i \leqslant n$, define coset i by $i = (i-1).a_{i-1}$. Then, since $x \in K$, $n.a_n = x.1$. From the first relation we deduce the collapse $(k+1) = x^{-t} y.1$.

Partition the set $\{1, 2, \dots, n\}$ into subsets $\mathfrak{P}_j$, $1 \leqslant j \leqslant k$ where

$$\mathfrak{P}_j = \{x \in \mathbb{Z} \colon 1 \leqslant x \leqslant n, x = 1 + ik - (j-1)n, \quad \text{for some} \quad i \in \mathbb{Z}^+\}.$$

Then, given i, $1 \leqslant i \leqslant n$, $i \in \mathfrak{P}_j$ for some j, $1 \leqslant j \leqslant k$. Define $v(i)$ to be this j.

Since $(k, n) = 1$, given i such that $1 \leqslant i \leqslant n$ there exists a unique j, $0 \leqslant j \leqslant n-1$ with $i \equiv jk+1 \bmod n$. For the rest of this proof we shall assume that $jk+1$ has been reduced mod n. All integers representing cosets are reduced mod n and all integers j appearing in expressions $w'(j)$ or $w(j)$ are reduced mod n.

Define $w'(jk+1)$ for $0 \leqslant j \leqslant n-1$ by $w'(1) = 1$, $w'(k+1) = x^{-t}y$, and, by induction on j define

$$w'(jk+1) = x^{-u}w'((j-1)k+1)w'((j-2)k+1)^{-1}x^t w'((j-1)k+1).$$

Now consider the subgroup generator y from which we obtain the collapse $2k+1 = x^{-\delta-(u+t)}y^2.1$ where $\delta = v(2k+1)-v(k+1)$. Notice that

$$v(ik+1)-v((i-1)k+1) = 1$$

if $\lambda n \in \{(i-1)k+1, (i-1)k+2, \ldots, ik\}$ for some $\lambda \in \mathrm{N}$; otherwise

$$v(ik+1)-v((i-1)k+1) = 0.$$

From the $(k+1)$st relation

$$(a_{k+1}a_{k+2} \ldots a_k)^t(a_{k+1}a_{k+2} \ldots a_{2k}) = (a_{2k+1}a_{2k+2} \ldots a_{2k})^u(a_{2k+1}a_{2k+2} \ldots a_{3k}) = \alpha \text{ say},$$

we obtain

$$(k+1)((a_{k+1}a_{k+2} \ldots a_k)^t(a_{k+1}a_{k+2} \ldots a_{2k})) = x^{t+\delta}.(2k+1)$$

and

$$(2k+1)((a_{2k+1}a_{2k+2} \ldots a_{2k})^u(a_{2k+1}a_{2k+2} \ldots a_{3k})) = x^{u+\varepsilon}.(3k+1)$$

where δ is as above and $\varepsilon = v(3k+1)-v(2k+1)$.

Then

$$(3k+1) = x^{-(\varepsilon+u)}(2k+1).\alpha = x^{-(\delta+\varepsilon+u)}w'(2k+1).1.\alpha,$$

and

$$(k+1).\alpha = x^{t+\delta}(2k+1).$$

Using $(k+1) = w'(k+1).1$ and $(2k+1) = x^{-\delta}w'(2k+1).1$ we obtain the collapse

$$\begin{aligned}(3k+1) &= x^{-(\delta+\varepsilon+u)}w'(2k+1)(w'(k+1))^{-1}x^t w'(2k+1).1\\ &= x^{1-v(3k+1)}w'(3k+1).1.\end{aligned}$$

Using induction on i, we can deduce from the relation $((i-2)k+1)$ that

$$(ik+1) = x^{1-v(ik+1)}w'(ik+1).1.$$

Therefore, for any coset j, $1 \leqslant j \leqslant n$ we have $j = x^{1-v(j)}w'(j).1$. Define

$$w(j) = x^{1-v(j)}w'(j), \quad 1 \leqslant j \leqslant n.$$

Then, from $i.a_i = i+1$, $1 \leqslant i \leqslant n-1$ we deduce $w(i).1.a_i = w(i+1).1$; so $1.a_i = w(i)^{-1}\,w(i+1).1$, $1 \leqslant i \leqslant n-1$, and, similarly, $1.a_n = w(n)^{-1}x.1$.

Therefore $K = \mathrm{H}(r, n, s)$ and, using the modified Todd–Coxeter coset enumeration algorithm, relations yielding a presentation for K in terms of x and y arise from the two subgroup generators and the n relations for $\mathrm{H}(r, n, s)$. Both subgroup generators give rise to trivial relations. This is obvious for $x = a_1a_2 \ldots a_n$ and for the second generator $y = (a_{k+1}a_{k+2} \ldots a_k)^u\,(a_{k+1}a_{k+2} \ldots a_{2k})$ use

$$1.(a_{k+1}a_{k+2} \ldots a_k) = w(1+k)^{-1}xw(1+k).1$$

to obtain the relation

$$w(1+k)^{-1}x^u x^{v(2k+1)-v(k+1)}w(2k+1) = y.$$

This gives $w'(2k+1) = x^{-u}w'(k+1)y$, which is already known to be satisfied.

Now $1.(a_i a_{i+1} \dots a_{i-1}) = w(i)^{-1}xw(i).1$ and

$$1.(a_{i+k}a_{i+k+1} \dots a_{i+k-1}) = w(i+k)^{-1}xw(i+k).1.$$

Provided $i+k-1 \neq n$ and $i+2k-1 \neq n$ we obtain from the ith relation

$$x^{-u}w(i+k)w(i)^{-1}x^t x^{v(i+k)-v(i)}w(i+k) = x^{v(i+2k)-v(i+k)}w(i+2k).$$

This is easily seen to be satisfied on using the definition for w and for w'.

From relation $(n-2k+1)$ we obtain

$$w(n-2k+1)^{-1}x^t x^{v(n-k+1)-v(n-2k+1)}w(n-k+1) = w(n-k+1)^{-1}x^{u+1}$$

giving

$$x^{-u-1}w(n-k+1)w(n-2k+1)^{-1}x^t x^{v(n-k+1)-v(n-2k+1)}w(n-k+1) = 1. \tag{2}$$

From relation $(n-k+1)$ we obtain

$$w(n-k+1)^{-1}x^{t+1} = x^u x^{-t}y. \tag{3}$$

Using (3), (2) becomes

$$w(n-2k+1) = x^{2t+1}x^{v(n-k+1)-v(n-2k+1)}y^{-1}x^{2t-2u}y^{-1}x^{t-u}. \tag{4}$$

Now, replacing w by w' and using $v(n-k+1) = k$, (3) and (4) become

$$w'(n-k+1) = x^{t+k}y^{-1}x^{t-u} \tag{5}$$

and

$$w'(n-2k+1) = x^{2t+k}y^{-1}x^{2t-2u}y^{-1}x^{t-u}. \tag{6}$$

By induction on i it is straightforward to check that

$$w'(ik+1) = x^{-(i-1)u-t}yx^{(i-2)(t-u)}yx^{(i-3)(t-u)} \dots yx^{(t-u)}y^2. \tag{7}$$

From (7) we obtain

$$w'(n-2k+1) = x^{-(n-3)t-t}y^{-1}x^{(n-2)u+t}w'(n-k+1). \tag{8}$$

Subsituting into (8) the expressions for $w'(n-2k+1)$ and $w'(n-k+1)$ given by (5) and (6) we obtain

$$y^{-1}x^s y = x^r \tag{9}$$

and, from (5) and (7) with $i = n-1$, we obtain

$$x^{u-t}yx^{-k-nt}x^{(n-2)(t-u)}yx^{(n-3)(t-u)} \dots x^{(t-u)}y^2 = 1. \tag{10}$$

So the group $\mathrm{H}(r, n, s)$ has a presentation on the two generators x and y with the two relations (9) and (10). This proves the theorem.

THEOREM 3. *Suppose that $r \equiv s$ mod n and r is coprime to n; then*

(i) *if r is not coprime to s, $\mathrm{H}(r, n, s)$ is infinite,*

and

(ii) *if r is coprime to s, $\mathrm{H}(r, n, s)$ is metacyclic of order $r^n - s^n$, being an extension of a cyclic group of order $(r^n - s^n)/n$ by a cyclic group of order n.*

Proof. (i) Let $d = (r, s)$. Consider the homomorphic image J of $H(r, n, s)$ obtained by adding the relation $x^d = 1$ to the presentation for $H(r, n, s)$ given in Theorem 2. It is easy to see that $J = \langle x, y \mid x^d = 1, y^n = 1\rangle$ and so is infinite.

(ii) The relation $y^{-1}x^s y = x^r$ gives for any $\lambda, \mu, l, m \in \mathbb{Z}$,

$$x^{\lambda s^l} y^l = y^l x^{\lambda r^l} \quad \text{and} \quad y^m x^{\mu r^m} = x^{\mu s^m} y^m.$$

Combining these,

$$y^m x^{\mu r^m + \lambda s^l} y^l = x^{\mu s^m} y^{m+l} x^{\lambda r^l}.$$

But r^m is coprime to s^l; so, given any $\gamma \in \mathbb{Z}\setminus\{0\}$ we can find $\lambda, \mu \in \mathbb{Z}$ such that $\mu r^m + \lambda s^l = \gamma$.

This formula allows the y's to be collected in (10) of Theorem 2 to give a relation of the form $y^n = x^\eta$, for some $\eta \in \mathbb{Z}$. Hence y^n is central in $H(r, n, s)$. Since $y^{-n}x^{s^n}y^n = x^{r^n}$, we obtain $x^{r^n - s^n} = 1$.

But s is coprime to $r^n - s^n$ since $(r, s) = 1$. Hence $\exists\, \alpha, \beta \in \mathbb{Z}$ such that $\alpha s - \beta(r^n - s^n) = 1$. But then $y^{-1}x^{\alpha s}y = x^{\alpha r}$ and using $x^{r^n - s^n} = 1$ we obtain

$$y^{-1}xy = x^{\alpha r}. \tag{11}$$

Formula (11) can be used to reduce formula (10) to the form $y^n = x^\eta$, where, after some elementary summation of series,

$$\eta = s(\alpha r)^{n-3} - \left(\frac{r-s}{n}\right)\left(\frac{1 - n(\alpha r)^{n-3} + n(\alpha r)^{n-2} - (\alpha r)^n}{(1-\alpha r)^2}\right).$$

It is easy to see that $x^{(1-\alpha r)\eta} = 1$ and that the order of x is $(r^n - s^n, (1-\alpha r)\eta)$. But

$$(1-\alpha r)\eta = s(\alpha r)^{n-3}(1-\alpha r) - \left(\frac{r-s}{n}\right)\left(\frac{1 - n(\alpha r)^{n-3} + n(\alpha r)^{n-2} - (\alpha r)^n}{(1-\alpha r)}\right)$$

$$= \frac{(s-r)}{n}\frac{(1-(\alpha r)^n)}{(1-\alpha r)} + (r-s)(\alpha r)^{n-3} + s(\alpha r)^{n-3}(1-\alpha r).$$

Using the fact that $\alpha s \equiv 1 \bmod r^n - s^n$ we obtain

$$(1-\alpha r)\eta \equiv \left(\frac{s-r}{n}\right)\frac{(1-(\alpha r)^n)}{(1-\alpha r)} \bmod r^n - s^n.$$

Hence the order of x is $\left(r^n - s^n, \dfrac{r-s}{n}\left(\dfrac{1-(\alpha r)^n}{1-\alpha r}\right)\right)$. The proof of the theorem is immediate when we have proved that the order of x is $(r^n - s^n)/n$.

$$\frac{(r-s)}{n}\frac{(1-(\alpha r)^n)}{(1-\alpha r)} = \frac{1}{n}\left(r + \alpha r^2 + \ldots + \alpha^{n-1}r^n - s - \alpha rs - \ldots - (\alpha r)^{n-1}s\right).$$

Substituting $\alpha s = 1 + \beta(r^n - s^n)$ in the right-hand side gives

$$\frac{1}{n}(\alpha^{n-1}r^n - s) - \beta r\frac{(r^n - s^n)}{n}\left(1 + (\alpha r) + \ldots + (\alpha r)^{n-2}\right).$$

But $r = s+\lambda n$ for some $\lambda \in \mathbb{Z}$; so we obtain

$$\left(\frac{1}{n}(\alpha^{n-1}r^n-s)-\beta r\frac{(r^n-s^n)}{n}\left(1+(\alpha s)+\ldots+(\alpha s)^{n-2}\right)\right) \bmod r^n-s^n$$

$$=\frac{1}{n}(\alpha^{n-1}r^n-s)-\beta r\frac{(r^n-s^n)}{n}\left(\frac{1-(\alpha s)^{n-1}}{1-\alpha s}\right).$$

Replacing αs by $1+\beta(r^n-s^n)$ in the above expression, it reduces to

$$\left(\frac{1}{n}(\alpha^{n-1}r^n-s)-\beta r\frac{(r^n-s^n)}{n}(n-1)\right) \bmod r^n-s^n$$

$$\equiv\left(\frac{1}{n}(\alpha^{n-1}r^n-s)+\beta r\frac{(r^n-s^n)}{n}\right) \bmod r^n-s^n.$$

However s is coprime to r^n-s^n and so the h.c.f. remains unchanged on multiplying through by s^{n-1}. We obtain

$$\frac{1}{n}(r^n-s^n)+\frac{r^n}{n}\left((1+\beta(r^n-s^n))^{n-1}-1\right)+\frac{\beta r s^{n-1}}{n}(r^n-s^n)=\frac{1}{n}(r^n-s^n) \bmod r^n-s^n.$$

Certainly the class $H(r, n, s)$ contains metacyclic groups other than those given in Theorem 3. We conjecture that the groups $H(r, 4, 2)$, r odd, are metacyclic. We have found a 2-generator, 2-relation presentation for $H(r, 4, 2)$, r odd, namely

$$\langle x, y \mid (xyxyx)^{(r-1)/2} = yx^{-1},\quad yxy = (xyx^2)^2\rangle.$$

We have proved the conjecture for $r \leqslant 23$ and have shown further that for $r = 3, 5, 7, 9$ and 11, $H(r, 4, 2)$ has order 5, 15, 125, 1015 and 4905 respectively. However we have been unable to prove the conjecture in general.

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Mathematical Institute,
University of St. Andrews,
St. Andrews, KY16 9SS,
Scotland.

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ON A GROUP PRESENTATION DUE TO FOX

BY
C. M. CAMPBELL AND E. F. ROBERTSON

In 1956 R. H. Fox had occasion, while investigating fundamental groups of topological surfaces, to believe that the group $\langle a, b \mid ab^2=b^3a, ba^2=a^3b\rangle$ was trivial. Using the Todd-Coxeter coset enumeration algorithm a proof was obtained, see [3], and this algorithmic proof was used to produce an algebraic proof, see [2]. In [1] Benson and Mendelsohn, using a similar method to that of [2] showed that $\langle a, b \mid ab^n=b^{n+1}a, ba^n=a^{n+1}b\rangle$ is trivial. In this note we give a direct proof for the more general problem of describing the structure of the group $\langle a, b \mid ab^n=b^\ell a, ba^n=a^\ell b\rangle$.

We use $|\cdot|$ to denote the order of a group, the order of a subgroup and the modulus of an integer, the context making it clear which is intended.

THEOREM. *Let $G=\langle a, b \mid ab^n=b^\ell a, ba^n=a^\ell b\rangle$. Then if*

(i) *$(n, \ell)\neq 1$, G is infinite;*

(ii) *$(n, \ell)=1$, G is metacyclic of order $|\ell-n|^3$.*

Proof. We can assume without loss of generality that $n\leq \ell$.

(i) If $(n, \ell)=d\neq 1$ then adding the relations $a^d=b^d=1$ to G shows that $\mathbb{Z}_d * \mathbb{Z}_d$, the free product of two copies of the cyclic group of order d, is a homomorphic image of G. Therefore G is infinite.

(ii) Assume $(n, \ell)=1$. The relation $ab^na^{-1}=b^\ell$ gives, for any i,

$$a^ib^{n^i}a^{-i}=b^{\ell^i}. \tag{1}$$

Putting $i=n$ in (1) and conjugating by b^{-1} we obtain $ba^nb^{n^n}a^{-n}b^{-1}=b^{\ell^n}$ and so

$$a^\ell b^{n^n}a^{-\ell}=b^{\ell^n}. \tag{2}$$

However (1) with $i=\ell$ is $a^\ell b^{n^\ell}a^{-\ell}=b^{\ell^\ell}$ and thus $b^{\ell^n(\ell^{\ell-n}-n^{\ell-n})}=1$. Raising (2) to the power $\ell^{\ell-n}-n^{\ell-n}$ we obtain $b^{(\ell^{\ell-n}-n^{\ell-n})}=1$, since ℓ and n are coprime.

Now $(n, \ell^{\ell-n}-n^{\ell-n})=1$ so there exist integers α, β such that $\alpha n+\beta(\ell^{\ell-n}-n^{\ell-n})=1$. Then $G\cong\langle a, b \mid aba^{-1}=b^{\alpha\ell}, bab^{-1}=a^{\alpha\ell}, a^{(\ell^{\ell-n}-n^{\ell-n})}=b^{(\ell^{\ell-n}-n^{\ell-n})}=1\rangle$. It is easy to see that the order of a and b is

$$\begin{aligned}
&((\alpha\ell-1)^2, (\ell^{\ell-n}-n^{\ell-n}))\\
&\qquad=(\alpha^2\ell^2-2\alpha\ell+1, (\ell^{\ell-n}-n^{\ell-n}))\\
&\qquad=(n^2\alpha^2\ell^2-2n^2\alpha\ell+n^2, (\ell^{\ell-n}-n^{\ell-n})) \quad \text{since} \quad (n^2, (\ell^{\ell-n}-n^{\ell-n}))=1.\\
&\qquad=((\ell-n)^2, (\ell^{\ell-n}-n^{\ell-n}))=(\ell-n)^2.
\end{aligned}$$

Now $aba^{-1}b^{-1}=b^{\alpha\ell-1}$ so $a^{1-\alpha\ell}=b^{\alpha\ell-1}$. Raising this to the power n gives $a^{n-\alpha n\ell}=b^{\alpha n\ell-n}$ showing that $a^{n-\ell}=b^{\ell-n}$.

Therefore $\langle a\rangle$ is normal in G, $|G/\langle a\rangle|=\ell-n$ and $|\langle a\rangle|=(\ell-n)^2$ giving the result.

COROLLARY. *The group* $\langle a, b \mid ab^n=b^{n+1}a,\ ba^n=a^{n+1}b\rangle$ *is trivial.*

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MATHEMATICAL INSTITUTE,
UNIVERSITY OF ST. ANDREWS,
ST. ANDREWS, KY16 9SS,
SCOTLAND

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